Grad QM PSet 1 - Some Linear Algebra

I recommend reading chapter 1 of Weinberg's Lectures on Quantum Mechanics for some interesting historical background.

If you use quantum mechanics in practice, one of the most common situations you will encounter is one where you can solve a very simple QM system exactly, but the system that you actually care about differs from the simple system by a small change. This is called perturbation theory. In this problem we will study a very relevant analogy in plain linear algebra. For convenience we will always assume that eigenvalues λ_i are ordered with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ for an $n \times n$ matrix.

1. Recall the relationship between the trace of a matrix, the determinant of a matrix, and the matrix's eigenvalues. Explain why $\log(\det(M)) = \text{ Tr } (\log(M))$.
2. Argue that the largest eigenvalue of an Hermitian matrix M is given by the maximum of $v^\dagger M v$ over all vectors v that have norm 1, ie $|v| = 1$.
3. Now say we have an Hermitian matrix M that differs from an Hermitian matrix H by a 'small' matrix Δ, so $H = M + \Delta$. Using the last step, what inequality can you derive relating the largest eigenvalues of each of M, H, Δ?

A subspace W of a vector space V is a subset of vectors in V that form a proper vector space. Note that W has to be a full vector space, so it must be *closed* under vector linear combinations, must contain the 0 vector, and must contain $-w$ if $w \in W$. It turns out that we can characterize eigenvalues using min-max statements involving subspaces. This can be used to extend the inequality you just derived to all eigenvalues.

1. Show that λ_i, the ith largest eigenvalue of an Hermitian matrix M acting in a vector space V, can be characterized as

$$\lambda_i(M) = \max_{\dim(W)=i} \left[\min_{w \in W, |w|=1} w^* M w \right] \tag{1}$$

where the first maximum is over all *subspaces* $W \subset V$ with fixed dimension; the minimum is just over all vectors w in this subspace W. Equivalently the same logic shows that

$$\lambda_i(M) = \min_{\dim(W)=n-i+1} \left[\max_{w \in W, |w|=1} w^* M w \right] \tag{2}$$

2. Now use this to prove the Weyl inequalities

$$\lambda_{i+j-1}(M + \Delta) \leq \lambda_i(M) + \lambda_j(\Delta) \tag{3}$$

Note in particular that if $j = 1$ we constrain $\lambda_i(M + \Delta)$ in terms of $\lambda_i(M)$ and the largest eigenvalue of Δ. In QMs, we will be interested in this situation when $M + \Delta$ is the Hamiltonian operator.

If you have trouble with these last two, feel free to look for resources online.