QFT Problem Set 5 - Due Nov. 10

You should read chapters 8-13 by the end of the semester (most of which are very nice), although we may not cover it all completely. As usual, *problems* are extra credit seekers, although everyone should look at them.

1. **Book Problems** 10.1, *10.5*, *11.1*

2. **Maxwell Chern-Simons Theory** Consider a theory in $2+1$ dimensions with action

$$S = \int d^3x \left(-\frac{1}{4} F_{\mu\nu}^2 + \lambda \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right) \tag{1}$$

The second term is called a Chern-Simons term. What is its scaling dimension? *How does it transform under C,P,T?*

Choose a gauge, for example the axial gauge $A_2 = 0$, eliminate any non-propagating degrees of freedom using their equations of motion, and obtain a classical action for the remaining degrees of freedom. How many propagating modes are there? What is their dispersion relation (relation between energy and momentum)? What is the effect of the Chern-Simons term at long distances?

You can learn more about the Chern-Simons theory, and how it can be used to explain the fractional quantum hall effect and the existence of particles with fractional statistics, in Zee's QFT in a Nutshell, chapter VI.1.

3. **Using $\epsilon^{\mu\nu\rho\sigma}$ in 4d** Consider supplementing the photon Lagrangian with another dimensionless term, so we have

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 + \theta \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right) \tag{2}$$

What is this new operator when written in terms of $\vec{E}$ and $\vec{B}$? Show that it does not contribute to the Feynman rules when θ is a constant. However, you should find that the operator

$$\frac{\pi(x)}{\Lambda} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{3}$$

does contribute to the Feynman rules. In fact, this is effective operator that allows the neutral pion to decay into 2 photons; its origin as a dimension 5 operator helps to explain why the neutral pion is relatively long-lived. Derive the Feynman rule for this interaction.

Now consider the terms involving fermion bilinears

$$\frac{1}{\Lambda} F_{\mu\nu} \bar{\psi} \gamma^\mu \gamma^\nu \psi, \quad \frac{1}{\Lambda} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} \bar{\psi} \gamma_\mu \gamma_\nu \psi \tag{4}$$

What's the physical interpretation of these two terms (what are they in terms of $\vec{E}$ and $\vec{B}$, and how do they affect the Dirac equation)? How do they differ? *How do they transform under C,P,T?*

4. **Selivanov's Method for Classical/Tree Level Scattering Amplitudes** There is a method for computing tree-level scattering amplitudes that makes it clear that they are a purely classical phenomenon associated with the scattering of plane waves. As far as I know this method has only been used in a few obscure papers by K. G. Selivanov (the only case where it's more useful than Feynman diagrams is for certain polarization states of gauge bosons and gravitons).

   The idea is that the scattering amplitude is simply the action, or the exponent in the path integral iS, evaluated on a certain solution of the classical equations of motion. That is

   $$i(2\pi)^4\delta^4(\Sigma_i p_i)\mathcal{M}_{tree} = iS[\phi_{cl}(x)] \tag{5}$$

   The solution ϕ_{cl} is written as

   $$\phi_{cl}(x) = \sum_i \epsilon_i e^{ip_i\cdot x} + \phi_{cl}^{(1)} + \phi_{cl}^{(2)} + \cdots \tag{6}$$

   where the variable ϵ_i just keeps track of a plane wave for the ith, which has momentum p_i. The higher order terms in $\phi_{cl}^{(j)}$ are computed by using the first term as a source and solving the classical equations of motion order by order in the coupling constant, as we discussed early on in class. Then one evaluates $iS[\phi_{cl}]$ and looks at the term proportional to the product of the ϵ_i, that is

   $$iS[\phi_{cl}(x)] \supset i(2\pi)^4\delta^4(\Sigma_i p_i)\mathcal{M}_{tree}\prod_{i=1}^{n}\epsilon_n \tag{7}$$

   Thus by looking at the term proportional to the product of the ϵ_i, where each occurs only once, one can extract the tree level scattering amplitude $\mathcal{M}_{tree}$. Let's see how this works in two examples.

   First consider a theory with action

   $$S[\phi] = \int d^4x\left[-\frac{1}{2}(\phi\Box\phi) - \frac{\lambda}{4!}\phi^4\right] = \int d^4x\left[-\frac{1}{2}\phi\left(\Box\phi + \frac{\lambda}{6}\phi^3\right) + \frac{\lambda}{4!}\phi^4\right] \tag{8}$$

   In the second way of writing the action, which terms contribute when we evaluate $S[\phi]$ on a solution to the classical equations of motion? Let's compute the 2-to-2 amplitude, so

   $$\phi_{cl}^{(0)}(x) = \epsilon_1 e^{ip_1 x} + \epsilon_2 e^{ip_2 x} + \epsilon_3 e^{ip_3 x} + \epsilon_4 e^{ip_4 x} \tag{9}$$

   If you're clever, you can already plug this into the action to get the scattering amplitude. If you're not, then you should write down a solution for $\phi^{(1)}$ and then compute the tree level scattering amplitude.

   Now consider the theory

   $$S = \int d^4x\left[-\frac{1}{2}(\phi\Box\phi) - \frac{g}{3!}\phi^3\right] = \int d^4x\left[-\frac{1}{2}\phi\left(\Box\phi + \frac{g}{2}\phi^2\right) + \frac{g}{12}\phi^3\right] \tag{10}$$

Use the same $\phi_{cl}^{(0)}$ to compute the $\phi_{cl}^{(1)}$ you need to get the 2-to-2 scattering amplitude. Do you need $\phi_{cl}^{(2)}$? Plug your ϕ_{cl} into the action to get the momentum space scattering amplitude $\mathcal{M}_{tree}$ via this method.

Can you explain in words why Selivanov's method works? Can you prove it? I don't think anyone's written down a pedagogical proof, so if you do you might even be able to publish it.