

QFT Problem Set 1 - Due Sept. 15, 2015

You should read the first two chapters of the book. The problems surrounded by stars, e.g. *2.5*, are extra credit (but you should definitely read them and think about them a bit). Graduate students in theoretical physics who are taking QFT for the first time should attempt these problems.

1. **Units** Show that we can convert between energy, momentum, inverse length, and inverse time by multiplying by various factors of \hbar and c . Thus these dimensionful quantities are all directly comparable once we choose units where $\hbar = c = 1$. What is a length scale and time scale associated with one $\text{GeV} = 10^9 \text{ eV}$? What are the units of the Newton constant of gravity, and what is its value expressed in terms of powers of GeV ?
2. **Textbook Problems:** 2.1, 2.3, 2.4, 2.6, *2.5*
3. **Asymptotic Series and a Toy Integral** You are probably most familiar with series (e.g. Taylor or Laurent) that converge. In QFT we will spend much of the course studying series expansions that do not converge, but that are useful anyway. These are called *asymptotic series*, and there is a rich and fascinating mathematical literature on their behavior.

Compute the integral

$$I_0 = \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma}} \quad (1)$$

in some nice way, e.g. by squaring it and then being clever.

Now consider the more general integral

$$I(\lambda) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2} - \frac{\lambda}{4}x^4} \quad (2)$$

If we imagine a situation where λ is a very small number, then it makes sense to try to expand the integral in a power series in λ . Show that if we expand the integrand to n th order in λ and perform the integral we find a series

$$I(\lambda) \rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2n + \frac{1}{2})}{n!} \lambda^n \quad (3)$$

What is the radius of convergence of this series in λ ? Now evaluate the integral (e.g. in Mathematica) at $\lambda = 0.02$ and compare it to the sum of the first M terms, for $M = 0, 1, 2, \dots, 30$ (a good strategy is to plot the logarithm of the difference between the exact result and successive terms in the series). What do you notice, and what happens as you change the value of λ ? How many terms should we keep to get the best possible approximation for a given value of $\lambda \ll 1$?

The integral you've studied is a toy version of the path integrals we encounter in QM and QFT; here λ plays the role of \hbar . We will see that perturbation theory in QFT provides an asymptotic series expansion directly analogous to what you've just found.

4. **Balls and Springs in More Dimensions** In class we studied a 1d lattice. Consider a square lattice of balls and springs in 2 space dimensions, with a Lagrangian¹

$$L = \sum_{v,h=1}^N \left[\frac{m}{2} \dot{\phi}_{v,h}^2 - \frac{m}{2} c_V^2 \left(\frac{\phi_{v,h} - \phi_{v+1,h}}{a} \right)^2 - \frac{m}{2} c_H^2 \left(\frac{\phi_{v,h} - \phi_{v,h+1}}{a} \right)^2 \right] \quad (4)$$

Compute the equations of motion and diagonalize them with a Fourier transform in both the horizontal and vertical coordinate directions. You should find a spectrum of decoupled harmonic oscillators, labeled by horizontal and vertical momenta. What is the dispersion relation? Now consider the continuum limit, where $a \rightarrow 0$. What is the meaning of the constants c_V and c_H , and what is special about $c_V = c_H$?

5. ***A Thin Layer of Atoms as an Extra Dimension*** Now consider the same setup as above, except with only two lattice sites in the vertical direction (but $N \gg 1$ in the horizontal direction). Phrased purely in terms of the horizontal direction and horizontal momenta, what does the spectrum look like? You should find more than one type of wave propagating in the horizontal direction, with the massive relativistic dispersion relation $\omega^2 = m^2 + c_h^2 k^2$ for some values of m which you should compute, including one mode with $m = 0$. What would happen if we had 3, 4, or k lattice sites in the vertical direction? This is an elementary example where we have a ‘small extra dimension’ and a ‘Kaluza-Klein tower of states’.

6. ***Classical Limit of Canonical Quantization*** Given a classical system, we are confronted with a very general problem – how can we find a quantum mechanical system that turns into this classical system in ‘the classical limit’, when $\hbar \rightarrow 0$? There is no completely general solution to this problem. However, for virtually all examples of relevance to modern physics, there is a standard way to proceed, called ‘Canonical Quantization’. You can consult Weinberg’s Quantum Mechanics, chapter 9, to read about the general story.

Let’s consider the classical limit of a canonically quantized classical system. So consider a quantum harmonic oscillator, i.e. the system with $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$. Now consider a state

$$\psi(x_0, k_0; x) = \sqrt{\frac{1}{\pi\sigma^2}} e^{-\frac{(x-x_0)^2}{2\sigma^2} + ik_0 x/\hbar} \quad (5)$$

What are the expectation values and variances of x and p in this state? Show that the commutator $[F(\hat{x}, \hat{p}), G(\hat{x}, \hat{p})]$ for general functions F and G of the operators \hat{x} and \hat{p} is proportional to the Poisson bracket $\{F, G\}_{x,p}$. Note how this can be generalized whenever the canonical commutation relations apply. Find a limit where we can simply view $F(\hat{x}, \hat{p})$ as $F(\langle x \rangle, \langle p \rangle)$ so that we recover classical mechanics.

¹In the real world we would want ϕ to be a vector quantity, encoding both the magnitude and direction of the displacement, but here we have chosen to make $\phi_{v,h}$ a scalar quantity for simplicity – so it doesn’t truly describe the displacement of the atoms.