CHARACTERISTIC CYCLE OF A CONSTRUCTIBLE SHEAF

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1. INTRODUCTION

Let \( k \) be a field and \( X/k \) a smooth variety. Let \( \Lambda \) be a finite field of characteristic \( l \) coprime to \( \text{char}(k) \). \( F \) will be a constructible complex of \( \Lambda \)-modules on \( X \).

Associated to \( F \) is the support \( \text{Supp}(F) \subset X \). The support doesn’t contain much information so instead we will work with the cotangent bundle \( T^*X \), viewed as the total space of the rank \( n \) locally free sheaf \( \Omega^1_X \). In particular \( T^*X \) is a \( 2n \)-dimensional smooth variety.

**Definition 1.1.** (Beilinson) The singular support \( \text{SS}(F) \subset T^*X \) is a closed conical subset associated to \( F \) such that \( \text{SS}(F) = \bigcup C_\alpha \) for \( C_\alpha \) \( n \)-dimensional irreducible components.

Recall that a subset \( S \subset T^*X \) is conical if it is stable under the \( \mathbb{G}_m \)-action scaling the fibers. While \( \text{SS}(F) \) carries more information than \( \text{Supp}(F) \), its only a subset. We want to enhance it into a cycle.

**Definition 1.2.** The characteristic cycle \( \text{CC}(F) = \sum_\alpha m_\alpha C_\alpha \) for \( m_\alpha \in \mathbb{Z} \) where \( \text{SS}(F) = \bigcup C_\alpha \).

**Remark 1.3.** When \( F \) is a perverse sheaf, \( m_\alpha > 0 \) for all \( \alpha \).

1.1. Plan. The plan for today is as follows:
- classical example,
- direct image,
- characteristic class.

2. AN EXAMPLE

Let \( \dim X = 1 \). Then \( T^*X \) is a line bundle on a curve with projection \( T^*X \to X \). Any conical subset of \( T^*X \) must either be the zero section or a union of fibers. In this case, the singular support of a sheaf \( F \) on \( X \) is very explicit.

Let \( D \subset X \) be the minimal closed subset such that \( F|_{X \setminus D} \) is locally constant. Then

\[
\text{SS}(F) = T^*_X X \cup \bigcup_{z \in D} T^*_z X
\]

where \( T^*_X X \subset T^*X \) denotes the zero section. The characteristic cycle is given by

\[
\text{CC}(F) = -\left( \text{rk}(F|_{X \setminus D}) T^*_X X + \sum_{z \in D} (a_z F) T^*_z X \right)
\]

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where $a_z \mathcal{F}$ is the Artin conductor:

$$a_z \mathcal{F} := \text{rk}\mathcal{F}|_{X \setminus D} - \text{rk}\mathcal{F}_z + Sw_z \mathcal{F} \in \mathbb{Z}$$

and $Sw_z \mathcal{F}$ is the Swan conductor, a measure of the wild ramification.

**Remark 2.1.** When $\dim X = 1$, the singular support is always a Lagrangian subvariety of $T^*X$. This will not be true in general.

### 3. Direct Image

Let $f : X \to Y$ be a proper morphism of smooth varieties over $k$. Let $m = \dim Y$ and $\mathcal{F}$ on $X$ a constructible complex. Then the direct image $Rf_* \mathcal{F}$ is a constructible complex on $Y$. We want to study the relation between

$$\text{SS}(\mathcal{F}), \text{CC}(\mathcal{F})$$

in the cotangent bundle of $X$ and

$$\text{SS}(Rf_* \mathcal{F}), \text{CC}(Rf_* \mathcal{F})$$

in the cotangent bundle of $Y$.

Recall that pullback of differential forms gives a natural diagram

$$T^*X \leftarrow X \times_Y T^*Y \longrightarrow T^*Y$$

where the map on the right is proper. Then given a closed conical subset $C \subset T^*X$, we can construct $f_0(C)$ as follows:

$$T^*X \leftarrow a X \times_Y T^*Y \longleftarrow \longrightarrow T^*Y$$

where $f_0(C) := b((a^{-1}(C))$ is a closed conical subset since $b$ is proper.

**Theorem 3.1.** (Beilinson) $\text{SS}(Rf_* \mathcal{F}) \subset f_0(\text{SS}(\mathcal{F}))$.

We’d like to understand how to relate the characteristic cycle. We do this by using the intersection theoretic pullback and pushforward on Chow groups. Recall the Chow group $\text{CH}_m(f_0(\text{SS}(\mathcal{F})))$ is the group of algebraic cycles of dimension $m$ modulo rational equivalence.

Using pullback and pushforward on Chow groups we get a class $f_0 \text{CC}(\mathcal{F}) \in \text{CH}_m(f_0(\text{SS}(\mathcal{F})))$.

**Fact 3.2.** If $\dim f_0(\text{SS}(\mathcal{F})) \leq m = \dim Y$, then this is well defined on the level of algebraic cycles $\text{Z}_m(f_0(\text{SS}(\mathcal{F})))$ without rational equivalence.

**Conjecture 1.** Suppose $f : X \to Y$ is proper. Then $\text{CC}(Rf_* \mathcal{F}) = f_0 \text{CC}(\mathcal{F})$ in $\text{CH}_m(f_0(\text{SS}(\mathcal{F})))$.

**Theorem 3.3.** Assume $X$ and $Y$ are projective, $f : X \to Y$ is a projective morphism and $\dim f_0(\text{SS}(\mathcal{F})) \leq m$. Then $\text{CC}(Rf_* \mathcal{F}) = f_0 \text{CC}(\mathcal{F})$ as algebraic cycles.
Example 3.4. Let \( Y = \text{Spec} k \). Then we can compute the Euler-Poincaré characteristic using the characteristic cycle as

\[
\chi(X, \mathcal{F}) = (\text{CC}(\mathcal{F}), T^*X)_{T^*X}
\]

where the right hand side is the intersection product with the zero section. This is the index formula.

In the special case that \( \dim X = 1 \), putting this together with the computation of \( \text{CC}(\mathcal{F}) \) above we recover the classical Grothendieck-Ogg-Shafarevich formula. The general case is obtained from GOS by induction on \( \dim X \).

Example 3.5. Let \( \dim Y = 1 \). Then to understand \( \text{CC}(Rf_*\mathcal{F}) \), we need to compute the Artin conductors \( a_y Rf_*\mathcal{F} \) and these are given by

\[
a_y Rf_*\mathcal{F} = (\text{CC}(\mathcal{F}), df)_{T^*X, X_y}
\]

where the right hand side is an intersection product on \( T^*X \) supported over the fiber \( X_y \).

The special case \( \mathcal{F} = \Lambda \) is the constant sheaf gives a formula conjectured by Bloch in the 80's.

4. Characteristic class

Let \( X \) be a not necessarily smooth variety with a closed immersion \( i : X \to M \) into a smooth variety \( M \) of dimension \( N \). We can consider the characteristic cycle

\[
\text{CC}(i_*\mathcal{F}) = \sum m_\alpha C_\alpha, \quad C_\alpha \subseteq X \times_M T^*M \subseteq T^*M.
\]

Let

\[
X \times_M T^*M \subseteq \mathbb{P}(X \times_M T^*M \oplus A^1_X)
\]

be the projective closure of \( X \times_M T^*M \) and \( \overline{C_\alpha} \) the closure of \( C_\alpha \) in \( \mathbb{P}(X \times_M T^*M \oplus A^1_X) \).

We consider the cycle

\[
\left[ \sum m_\alpha \overline{C_\alpha} \right] \in \text{CH}_N(\mathbb{P}(X \times_M T^*M \oplus A^1_X)).
\]

A standard computation in intersection theory gives that

\[
\text{CH}_N(\mathbb{P}(X \times_M T^*M \oplus A^1_X)) = \bigoplus_{q \in \mathbb{Z}} \text{CH}_q(X) =: \text{CH}_*(X)
\]

and the characteristic class \( \text{CC}_X(\mathcal{F}) := [\sum m_\alpha \overline{C_\alpha}] \in \text{CH}_*(X) \) is independent of \( M \).

Let \( K_0(X, \Lambda) \) be the Grothendieck group of constructible sheaves. Then the characteristic class \( \text{CC}_X \) gives a homomorphism

\[
\text{CC}_X : K_0(X, \Lambda) \to \text{CH}_*(X).
\]

That is, \( \text{CC}_X \) is additive.
4.1. **Grothendieck’s question.** During SGA5 (though it is only recorded in Récoltes et Semailles) Grothendieck asked the following question:

**Question 1.** Given \( f : X \to Y \) a proper map, is the diagram

\[
\begin{array}{ccc}
    K_0(X, \Lambda) & \xrightarrow{Cc_X} & CH_\bullet(X) \\
    Rf_* \downarrow & & \downarrow r_* \\
    K_0(Y, \Lambda) & \xrightarrow{Cc_Y} & CH_\bullet(X)
\end{array}
\]

commutative?

In fact the answer is NO and Grothendieck gave an example in Récoltes et Semailles. However, one can ask if the statement is true just for \( CH_0 \).

**Conjecture 2.** The diagram

\[
\begin{array}{ccc}
    K_0(X, \Lambda) & \xrightarrow{(Cc_X)_0} & CH_0(X) \\
    Rf_* \downarrow & & \downarrow r_* \\
    K_0(Y, \Lambda) & \xrightarrow{(Cc_Y)_0} & CH_0(X)
\end{array}
\]

is commutative.

**Theorem 4.1.** Conjecture 1 implies Conjecture 2.

**Theorem 4.2.** (Umezaki-Yang-Zhao) Let \( X \) and \( Y \) be projective varieties over a finite field \( k \) and \( f : X \to Y \) a projective morphism. Then Conjecture 2 is true.