Introduction to Mixed Modular Motives

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Acknowledgments

Antecedents and Inspiration

- Beilinson and Levin: elliptic polylogarithms
- Yuri Manin: iterated integrals of modular forms;
- Francis Brown: multiple modular values (and motives)
- The many who contributed to the mixed Tate motive story (to be named later . . . ), which I will review shortly.

Collaborators

- Francis Brown (ongoing)
- Makoto Matsumoto (universal mixed elliptic motives)
\( \mathcal{M}_{1,1} \) is the moduli stack of elliptic curves. It has two manifestations of concern to us:

1. the analytic orbifold:

\[
\mathcal{M}^\text{an}_{1,1} = \frac{\text{SL}_2(\mathbb{Z})}{\mathfrak{h}},
\]

where \( \mathfrak{h} \) denotes the upper half plane \( \text{Im} \tau > 0 \).

2. the algebraic stack over \( \mathbb{Q} \):

\[
\mathcal{M}_{1,1}/\mathbb{Q} = \frac{\mathbb{G}_m}{(\mathbb{A}^2_{\mathbb{Q}} - \{u^3 - 27v^2 = 0\})},
\]

where \( t \cdot (u, v) = (t^4 u, t^6 v) \).
Each (normalized) Hecke eigen cusp form $f$ of $SL_2(\mathbb{Z})$ of weight $2n + 2$ determines a pure motive $V_f$ of weight $2n + 1$ and Hodge type $(2n + 1, 0), (0, 2n + 1)$. Its Hodge and $\ell$-adic Galois realizations occur as summands of the cuspidal cohomology group

$$H^1_{\text{cusp}}(SL_2(\mathbb{Z}), S^{2n}H) = H^1_{\text{cusp}}(\mathcal{M}_{1,1}, S^{2n}\mathbb{H}) = \bigoplus_{f \text{ eigen cusp form}} V_f$$

where $S^{2n}H$ is the $2n$th symmetric power of the standard representation and $S^{2n}\mathbb{H}$ is the corresponding local system.

The Eisenstein series $G_{2n+2}$ determines the motive $\mathbb{Q}(-2n - 1)$ of weight $4n + 2$:

$$H^1(\mathcal{M}_{1,1}, S^{2n}\mathbb{H}) = H^1_{\text{cusp}}(\mathcal{M}_{1,1}, S^{2n}\mathbb{H}) \oplus \mathbb{Q}(-2n - 1).$$
For distinct eigenforms $f_1, \ldots, f_m$, we then have the motive

$$V := S^{r_1} V_{f_1} \otimes \cdots \otimes S^{r_m} V_{f_m}$$

with its Hodge and $\ell$-adic Galois realizations.

There are expected regulator mappings

$$\text{Ext}^1_{\text{mot}}(\mathbb{Q}, V(d)) \to \text{Ext}^1_{\text{MHS}}(\mathbb{R}, V_\mathbb{R}(d))^{\mathcal{F}_\infty} \cong V_\mathbb{R}$$

$$\text{Ext}^1_{\text{mot}}(\mathbb{Q}, V(d)) \to H^1_f(G_\mathbb{Q}, V_\mathbb{Q}_\ell(d))$$

which are conjectured by Beilinson and Bloch–Kato to be isomorphisms after tensoring with $\mathbb{R}$ and $\mathbb{Q}_\ell$, respectively. Their conjectured dimensions are given by the order of vanishing of a convolution $L$-function at an appropriate negative integer.
The main goal of this talk is to address the question:

**Question:** Where does one find such extensions in nature, and what might their periods be?

I believe that the case $m = 1, r_1 = 1$ is well understood.

Many of the ideas in this talk for how to approach this are due to Francis Brown.
Iterated Integrals

Definition (K.-T. Chen)

Suppose that $M$ is a smooth manifold. For 1-forms $\omega_1, \ldots, \omega_r$ and a piecewise smooth path $\gamma : [0, 1] \rightarrow M$, define

$$
\int_{\gamma} \omega_1 \ldots \omega_r = \int_{\Delta^r} \omega_1 \times \omega_2 \times \cdots \times \omega_r
$$

$$
= \int \cdots \int f_1(t_1)f_2(t_2)\ldots f_r(t_r) \, dt_1 \, dt_2 \ldots \, dt_r
$$

where $\gamma^* \omega_j = f_j(t) \, dt$ and $\Delta^r$ is the time ordered $r$-simplex $0 \leq t_1 \leq t_2 \leq \cdots \leq t_r \leq 1$ in $[0, 1]^r$. 

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Properties: reflect combinatorial properties of simplices

1. **shuffle product:**

\[
\int_\gamma \omega_1 \ldots \omega_r \int_\gamma \omega_{r+1} \ldots \omega_{r+s} = \sum_{\sigma \in \text{sh}(r,s)} \int_\gamma \omega_{\sigma(1)} \omega_{\sigma(2)} \ldots \omega_{\sigma(r+s)}
\]

2. **coproduct:**

\[
\int_{\gamma \mu} \omega_1 \ldots \omega_n = \sum_{j=1}^n \int_\gamma \omega_1 \ldots \omega_j \int_\mu \omega_{j+1} \ldots \omega_n
\]

3. **antipode:**

\[
\int_{\gamma^{-1}} \omega_1 \ldots \omega_r = (-1)^r \int_\gamma \omega_r \ldots \omega_1
\]
Example: The $n$-logarithm:

$$\ln_n(z) := \sum_{k \geq 1} \frac{z^n}{kn} = \int_0^z \frac{dw}{1 - w} \left( \frac{dw}{w} \right)^{n-1}, \quad |z| < 1$$

and its special value $\ln_n(1) = \zeta(n)$ are iterated integrals.
Multiple Zeta Values

Definition

Suppose that $r \geq 1$ and that $n_1, \ldots, n_r$ are positive integers with $n_r > 1$. The *multiple zeta number* \( \zeta(n_1, \ldots, n_r) \) is defined by the convergent series

\[
\zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_r^{n_r}}.
\]

The integer $r$ is called the *depth*, and $n_1 + \cdots + n_r$ is called the *weight* of $\zeta(n_1, \ldots, n_r)$.

These satisfy lots of combinatorial identities. They are graded by weight and filtered by depth.
Let $\omega_0 = \frac{dw}{w}$ and $\omega_1 = \frac{dw}{1-w}$. The iterated integral $\int_0^1 \omega_{\epsilon_1} \ldots \omega_{\epsilon_r}$ converges if and only if $\epsilon_1 = 1$ and $\epsilon_r = 0$.

**Kontsevich formula**

$$
\zeta(n_1, \ldots, n_r) = \int_0^1 \omega_1^{n_1-1} \omega_0^{n_2-1} \omega_1^{n_r-1} $$

Divergent iterated integrals can be regularized provided that one chooses non-zero tangent vectors $\vec{v}_0$ at 0 and $\vec{v}_1$ at 1. The regularization of this iterated integral is also an MZV when $\vec{v}_0$ and $\vec{v}_1$ are $\pm \partial/\partial w$. 
Regularizing iterated integrals: trivial coefficients

Suppose that $X$ is a complex curve, $P, Q \in X$ and $\omega_1, \ldots, \omega_r$ are holomorphic 1-forms on $X' := X - \{P, Q\}$ with at worst logarithmic singularities at $P, Q$. (i.e., each $\omega_j$ is a differential of the 3rd kind.)

To regularize the possibly divergent $\int_P^Q \omega_1 \ldots \omega_r$ choose non-zero tangent vectors $\vec{v} \in T_P X$ and $\vec{w} \in T_Q X$. Choose holomorphic coordinates $s$ and $t$ on $X$ centered at $P, Q$, respectively, where $\vec{v} = \partial/\partial s$ and $\vec{w} = \partial/\partial t$. 
Set

\[ f(s, t) = \int_{P(s)}^{Q(t)} \omega_1 \ldots \omega_r. \]

Then the theory of ODEs with regular singular points implies

\[ f(s, t) = \sum_{j=0}^{r} \sum_{k=0}^{r} a_{j,k}(s, t)(\log s)^j(\log t)^k \]

where each \( a_{j,k}(s, t) \) is holomorphic near \((0, 0)\). Define

\[ \int_{\vec{v}} \omega_1 \ldots \omega_r := a_0(0). \]

It depends only on \( \vec{v} \) and \( \vec{w} \) and not on the choice of \( t \) and \( s \).
Regularization is compatible with properties (shuffle product, antipode, coproduct) of iterated integrals.

**Examples**

Take $z = \mathbb{C}$, $P = 0$, $Q = 1$, $\vec{v} = \partial / \partial w$ and $\vec{w} = -\partial / \partial w$.

1. Since $\int_t^1 \frac{dw}{w} = -\log t$, $\int_{\vec{v}}^{\vec{w}} \frac{dw}{w} = 0$. Similarly, $\int_{\vec{v}}^{\vec{w}} \frac{dw}{(1 - w)} = 0$.

2. Since $\int_0^1 \omega_1 \omega_0 = \zeta(2)$, the shuffle product formula

$$0 = \int_{\vec{v}}^{\vec{w}} \omega_0 \int_{\vec{v}}^{\vec{w}} \omega_1 = \int_{\vec{v}}^{\vec{w}} \omega_0 \omega_1 + \int_{\vec{v}}^{\vec{w}} \omega_1 \omega_0.$$ 

gives $\int_{\vec{v}}^{\vec{w}} \omega_0 \omega_1 = -\zeta(2)$. 

1. Manin considered iterated integrals of classical modular forms.

2. Brown considered regularized iterated integrals of classical modular forms, where the base point is the tangent vector $\partial/\partial q$ at the origin of the $q$-disk.

Such regularized iterated integrals will be regarded as periods of a suitable completion of

$$\pi_1(M_{1,1}, \partial/\partial q) \cong \text{SL}_2(\mathbb{Z}).$$

Later we will explain why $\pm \partial/\partial q$ are the most (only?) natural choices of base point.
Suppose that $f$ is a modular form of $SL_2(\mathbb{Z})$ of weight $2n + 2$. The corresponding cohomology class in $H^1(SL_2(\mathbb{Z}), S^{2n}H)$ is represented by the $SL_2(\mathbb{Z})$-invariant 1-form

$$\omega_f(b) := f(\tau)(b - \tau a)^{2n} d\tau.$$ 

where $a, b$ is a basis of the standard representation $H$ of $SL_2(\mathbb{Z})$.

Now suppose that $f_j, j = 1, \ldots, r$ are modular forms where $f_j$ has weight $2n_j + 2$. One considers the iterated integral

$$\int \omega_{f_1}(b_1) \omega_{f_2}(b_2) \cdots \omega_{f_r}(b_r)$$

which takes values in $S^{2n_1}H \otimes \cdots \otimes S^{2n_r}H$. 
Multiple modular values are numbers obtained by evaluating

$$\int \omega_{f_1}(b_1) \omega_{f_2}(b_2) \ldots \omega_{f_r}(b_r)$$

along the imaginary axis, which is a path in $\mathcal{M}_{1,1}$ from $\partial/\partial q$ to $-\partial/\partial q$, composed with a projection

$$S^{2n_1} H \otimes \ldots \otimes S^{2n_r} H \rightarrow \mathbb{C}$$

defined over $\mathbb{Q}$.

**Remark:** Regularized iterated integrals of Eisenstein series include all MZVs (RMH), but are more general (Brown: they include periods cusp forms). So multizeta values form a subring of mixed modular values.
The Tate motives over $\mathbb{Q}$ are $\mathbb{Q}(n)$, where $n \in \mathbb{Z}$.

**Examples:**

1. $\mathbb{Q}(0) = H_0(\text{point})$;
2. $\mathbb{Q}(1) = H_1(\mathbb{G}_m)$
3. $\mathbb{Q}(-1) = H^1(\mathbb{G}_m) \cong H^2(\mathbb{P}^1)$
4. $\mathbb{Q}(n) = \mathbb{Q}(1)^\otimes n$; Hodge type $(-n, -n)$.

**Realizations of $\mathbb{Q}(-1)$**

- **Betti:** $H^1(\mathbb{C}^*; \mathbb{Q})$
- **de Rham:** $H^1_{\text{DR}}(\mathbb{G}_m/\mathbb{Q}) = \mathbb{Q}\frac{dw}{w}$
- **$\ell$-adic:** $H^1_{\text{et}}(\mathbb{G}_m/\mathbb{Q}) \cong \mathbb{Q}_\ell$ on which $G_\mathbb{Q}$ acts by the inverse of the $\ell$-adic cyclotomic character.
Mixed Tate motives (MTMs) over $\mathbb{Q}$ are successive extensions of Tate motives. They have Betti, de Rham and $\ell$-adic realizations. The $\ell$-adic realization is $G_{\mathbb{Q}}$-module; Betti and de Rham define a mixed Hodge structure.

**Example:** $H_1(\mathbb{G}_m, \{1, x\}), x \in \mathbb{Q}^\times$

\[
0 \longrightarrow H_1(\mathbb{G}_m) \longrightarrow H_1(\mathbb{G}_m, \{1, x\}) \longrightarrow \tilde{H}_0(\{1, x\}) \longrightarrow 0
\]

\[
\begin{array}{ccc}
\mathbb{Q}(1) & \longrightarrow & \mathbb{Q}(0)
\end{array}
\]

This is unramified outside \{primes $p : \nu_p(x) \neq 0\}$.
The axioms of a neutral tannakian category over a char 0 field $F$ are what you obtain if you axiomatize the properties of the category of finite dimensional representation of a group over $F$. Essential features:

1. $F$-linear abelian category (internal Hom sets are vector spaces over $F$)
2. associative and commutative tensor product
3. trivial object and duals
4. compatibilities
5. faithful functor to $\text{Vec}_F$ (fiber functor)
Tannaka duality:

Every neutral tannakian category $\mathcal{C}$ over $F$ is equivalent to the category of representations of an affine group over $F$. It depends on the choice of a fiber functor $\omega$ and is denoted $\pi_1(\mathcal{C}, \omega)$. All are canonically isomorphic up to inner automorphisms.

Examples:

1. The category of semi-simple Tate motives over $\mathbb{Q}$ is tannakian. Fiber functors Betti or de Rham. The fundamental group is $\mathbb{G}_m/\mathbb{Q}$.

2. The category MHS of $\mathbb{Q}$-mixed Hodge structures.
Mixed Tate Motives

Theorem (Borel, Voevodsky, Levine, Deligne–Goncharov)

There is a \( \mathbb{Q} \)-linear tannakian category of MTMs unramified over \( \mathcal{O}_{K,S} \) and whose ext groups \( \text{Ext}_{\text{MTM}(\mathcal{O}_{K,S})}^j(\mathbb{Q}, \mathbb{Q}(n)) \) vanish when \( j > 1 \) and satisfy:

\[
\text{Ext}_{\text{MTM}(\mathcal{O}_{K,S})}^1(\mathbb{Q}, \mathbb{Q}(n)) \cong K_{2n-1}(\mathcal{O}_{K,S}) \otimes \mathbb{Q}.
\]

This implies that there is a natural isomorphism

\[
\pi_1(\text{MTM}(\mathbb{Z}), \omega^{\text{DR}}) \cong \mathbb{G}_m \ltimes \mathcal{K}
\]

where \( \mathcal{K} \) is a prounipotent group whose Lie algebra is the free Lie algebra

\[
\mathfrak{k} = \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots)^\wedge
\]
Suppose that $\Gamma$ is a discrete group and that $F$ is a field of characteristic zero. The unipotent completion $\Gamma^\text{un}/F$ of $\Gamma$ over $F$ is the fundamental group of the category of unipotent representations $\Gamma \to U(F)$ of $\Gamma$ defined over $F$. It is a prounipotent $F$-group. There is a canonical homomorphism $\Gamma \to \Gamma^\text{un}(F)$.

**Universal mapping property**

Given a representation $\rho : \Gamma \to U(F)$ of $\Gamma$, where $U$ is unipotent over $F$, there is a unique homomorphism $\Gamma^\text{un}/F \to U$ such that $\rho$ is the composite

$$
\Gamma \to \Gamma^\text{un}(F) \to U(F).
$$
Example: unipotent completion of a free group

If $\Gamma$ is the free group $\Gamma = \langle u, v \rangle$ and $ad - bc \neq 0$, then

$$\theta : F\Gamma^\wedge \xrightarrow{\sim} F\langle \langle X, Y \rangle \rangle$$

is a (complete) Hopf algebra isomorphism when

$$\theta(u) = \exp U \text{ and } \theta(v) = \exp V$$

where $U, V \in \mathbb{L}(X, Y)^\wedge$ and

$$U \equiv aX + bY \text{ and } V \equiv cX + dY \mod J^2.$$

Theta induces an isomorphism $\Gamma^\text{un}(F) \xrightarrow{\sim} \exp \mathbb{L}(X, Y)^\wedge$. 
Theorem (K.-T. Chen)

If $M$ is a smooth manifold and $x \in M$, then the coordinate ring $\mathcal{O}(\pi^\text{un}_1(M, x))$ is isomorphic to the set of “closed” iterated integrals on $M$.

Example: $M = \mathbb{P}^1 - \{0, 1, \infty\}$

The homomorphism $\Theta : \mathbb{C}\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, x) \to \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$

$$
\gamma \mapsto 1 + \int_\gamma \omega_0 e_0 + \int_\gamma \omega_1 e_1 + \int_\gamma \omega_0 \omega_0 e_0^2 + \int_\gamma \omega_0 \omega_1 e_0 e_1
$$

$$
+ \int_\gamma \omega_1 \omega_0 e_1 e_0 + \int_\gamma \omega_1 \omega_1 e_1^2 + \cdots
$$

produces an isomorphism $\mathcal{O}(\pi^\text{un}_1(\mathbb{P}^1 - \{0, 1, \infty\}, x))$ with the Hopf algebra of iterated integrals $\int \omega_{\epsilon_1} \ldots \omega_{\epsilon_r}$, where $\epsilon_j \in \{0, 1\}$. 
Question:
Where does one find extensions of mixed Tate motives in “nature”?

Following older work of Deligne (1987), Deligne and Goncharov proved:

**Theorem (Deligne–Goncharov, 2005)**

The coordinate ring of the unipotent completion of 
\[ \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v}) \] is an ind-object of \( \text{MTM}(\mathbb{Z}) \), where 
\[ \vec{v} = \partial / \partial w \in T_1 \mathbb{P}^1. \]
Every mixed Tate motive $V$ has a *Betti realization* $V^B$ and a *de Rham realization* $V^{\text{DR}}$. These are rational vector spaces and are isomorphic after tensoring with $\mathbb{C}$:

$$\text{comp} : V^{\text{DR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} V^B \otimes_{\mathbb{Q}} \mathbb{C}$$

The *periods* of $V$ are the complex numbers $\phi(\text{comp}(v))$, where $v \in V^{\text{DR}}$ and $\phi \in \text{Hom}_{\mathbb{Q}}(V^B, \mathbb{Q})$.

The periods of $\mathcal{O}(\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v})$ are precisely the multiple zeta values $\int_\gamma \omega_{\epsilon_1} \ldots \omega_{\epsilon_r}$.

**Question:** Are the periods of all objects of MTM($\mathbb{Z}$) multiple zeta values?
Brown’s Theorem

**Theorem (Brown 2012)**

The action of $\pi_1(\text{MTM}(\mathbb{Z}), \omega^{\text{DR}})$ on $\mathcal{O}(\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v}))$ is faithful.

**Corollary**

1. The periods of every object of $\text{MTM}(\mathbb{Z})$ are multiple zeta values.

2. The category $\text{MTM}(\mathbb{Z})$ is the smallest tannakian subcategory of $\text{MHS}_\mathbb{Q}$ that contains $\mathcal{O}(\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v}))$. 
While many of the main questions about mixed Tate motives and multiple zeta values have been resolved, there are important problems which remain. These include:

1. Understanding periods $\text{MTM}(\mathcal{O}_K, S)$ for other rings of $S$-integers, especially the case of $\mathbb{Z}[\mu_N, 1/N]$. Cf. work of Goncharov, Deligne on periods of $\pi_1^{\text{un}}(\mathbb{G}_m - \mu_N, \vec{v})$ and values of polylogarithms at roots of unity.

Where to next?

mixed Tate motives: $\mathbb{G}_m/\mathbb{Z}$

mixed elliptic motives: $E/K$

mixed modular motives: $\mathcal{M}_{1,1}/\mathbb{Z}$

motives associated to the universal elliptic curve: $\mathcal{E} \to \mathcal{M}_{1,1}$

**Issues:** We do not currently have a *tannakian* category of mixed motives with the correct exts in the elliptic or modular cases. The problem is that *Beilinson–Soulé vanishing* is not known except in the mixed Tate case.
Geography

\[
\begin{align*}
\zeta \left\{ 
\begin{array}{c}
M_{g,1} \\
q = 0 \quad \partial/\partial q \\
\text{identity section}
\end{array}
\right. \\
E_0 \quad E_{\partial/\partial q} \\
\partial/\partial w
\end{align*}
\]
Mixed Elliptic Motives

To each elliptic curve $E/K$, one can associate the pure motive $H^1(E)$ of weight 1. The category of elliptic motives associated to $E$ should be a tannakian category whose simple objects are quotients of the Tate twists $S^m H^1(E)(r)$ of symmetric powers of $H^1(E)$.

**Working hypothesis:** The coordinate ring $\mathcal{O}(\pi_{1}^{\text{un}}(E', \vec{w}))$ of $E' := E - \{\text{id}_E\}$ with basepoint a non-zero vector in $T_{\text{id} E}$ should be an object of this category. Could it generate it?

**Example:**
The smooth locus of the nodal cubic $E_0$ is $\mathbb{G}_m$. So $E'_0 = \mathbb{P}^1 - \{0, 1, \infty\}$ and the mixed elliptic motives associated to $E_0$ should be $\text{MTM}(\mathbb{Z})$. Further, one can show that $\pi_{1}^{\text{un}}(E_{\partial/\partial q}, \vec{w})$ is in $\text{MTM}(\mathbb{Z})$. 
It is useful to consider all elliptic curves at the same time. To this end we consider the universal elliptic curve $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$ and the local system $H = R^1\pi_*\mathbb{Q}$.

Elliptic motives in this situation are “motivic local systems” $\mathbb{V}$ over $\mathcal{M}_{1,1}$ whose weight graded quotients

$$W_r\mathbb{V}/W_{r-1}\mathbb{V}$$

are direct sums of the the simple local systems $S^mH(d)$.\(^1\) Here $W_*$ is the weight filtration

$$0 \subseteq \cdots \subseteq W_{r-1}\mathbb{V} \subseteq W_r\mathbb{V} \subseteq \cdots \subseteq \mathbb{V}.$$

\(^1\)Here one can take a motivic local system to be a compatible set of $\ell$-adic lisse sheaves and an admissible variation of MHS.
To rigidify the possible extensions, we insist that the fiber of such a $V$ over $\partial/\partial q$ be an object of $\text{MTM}(\mathbb{Z})$. These local systems are called **universal mixed elliptic motives**. Similarly one can define universal mixed elliptic motives over $\mathcal{M}_{1,2}$.

**Theorem**

*The local system over $\mathcal{M}_{1,2}$ whose fiber over $(E, \text{id}_E, x)$ is the Lie algebra of $\pi^\text{un}_1(E', x)$ is a (pro) universal mixed elliptic motive.*

Local systems $\nabla$ over $\mathcal{M}_{1,1}$ correspond to representations of

$$\pi_1(\mathcal{M}_{1,1}, \partial/\partial q) \cong \text{SL}_2(\mathbb{Z})$$

on their fiber $V := V_{\partial/\partial q}$. 
Relative Unipotent Completion

The standard representation of $SL_2(\mathbb{Q})$ is $H := H^1(E_{\partial/\partial q})$.

The category of finite dimensional representations of $SL_2(\mathbb{Z})$ that admit a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_N = V$$

with the property that each graded quotient $V_r/V_{r-1}$ is isomorphic to a direct sum of copies of $S^mH$ is tannakian over $\mathbb{Q}$.

**Definition**

The *relative (unipotent) completion* $\mathcal{G}^{rel}$ of $SL_2(\mathbb{Z})$ is the tannakian fundamental group of this category.
Properties of $G^{\text{rel}}$

1. There is a $\mathbb{Q}$-DR manifestation $G^{\text{rel}}_{\text{DR}}$ of $G^{\text{rel}}$ and a canonical isomorphism $G^{\text{rel}}_{\text{DR}} \times \mathbb{Q} \mathbb{C} \cong G^{\text{rel}} \times \mathbb{Q} \mathbb{C}$. So one can speak of periods of $G^{\text{rel}}$. All MMVs are periods.

2. The coordinate ring $\mathcal{O}(G^{\text{rel}})$ and the Lie algebra $g^{\text{rel}}$ have natural (ind and pro, resp.) MHS.

3. There is a natural action of $G_{\mathbb{Q}}$ on $G^{\text{rel}} \times \mathbb{Q} \mathbb{Q}_{\ell}$. It is unramified at all $p \neq \ell$ (Mochizuki–Tamagawa) and crystalline at $\ell$ (Olsson). (This is why we choose the base point to be $\pm \partial/\partial q$.)

4. It is a split extension $1 \to U^{\text{rel}} \to G^{\text{rel}} \to \text{SL}_2 \to 1$ where $U^{\text{rel}}$ is prounipotent. So $G^{\text{rel}} \cong \text{SL}(H) \ltimes U^{\text{rel}}$. (Levi)
The Lie algebra $u^{rel}$ of $U^{rel}$ is a free pronilpotent Lie algebra isomorphic to $\mathbb{L}(H_1(u^{rel}))^\wedge$.

There is a canonical $\text{SL}(H)$-invariant isomorphism of MHS

$$H_1(u^{rel}) \cong \prod_{n \geq 2} \left( S^{2n} H(2n + 1) \oplus \prod_{f \text{ eigenform} \atop \text{weight } 2n+2} V_f \otimes S^{2n} H(2n + 1) \right)$$

where $H = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)$. It is also $G_\mathbb{Q}$-equivariant after tensoring with $\mathbb{Q}_\ell$.

The weight graded quotients of $u^{rel}$ are therefore products of $V(d)$, where

$$V := S^{r_1} V_{f_1} \otimes \cdots \otimes S^{r_m} V_{f_m}.$$
Mixed Modular Motives

There should be a *tannakian* category of mixed motives, unramified over $\mathbb{Z}$ whose simple objects are $V(d)$, where

$$V = S^{r_1} V_{f_1} \otimes \cdots \otimes S^{r_m} V_{f_m}$$

where $f_1, \ldots, f_m$ are distinct eigenforms. We are far from having a construction of this category from Voevodsky motives.

Brown’s end run — motivated by his result for MTM($\mathbb{Z}$):

**Definition (Brown)**

Define the category $\text{MMM} = \text{MMM}(\mathbb{Z})$ of *mixed modular motives* unramified over $\mathbb{Z}$ to be the full tannakian subcategory of $\text{MHS}_{\mathbb{Q}}$ generated by $\mathcal{O}(\mathcal{G}^{\text{rel}})$.

Their periods include all *multiple modular values*. 

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Introduction to Mixed Modular Motives
There are “regulators”

$$\text{Ext}^1_{\text{MMM}}(\mathbb{Q}, V(d)) \rightarrow \text{Ext}^1_{\text{MHS}}(\mathbb{R}, V(d))^F$$

and

$$\text{Ext}^1_{\text{MMM}}(\mathbb{Q}, V(d)) \rightarrow H^1_f(G_{\mathbb{Q}}, V_{\mathbb{Q}_\ell}(d)).$$

One would hope that these are isomorphisms after tensoring with $\mathbb{R}$ and $\mathbb{Q}_\ell$, respectively. However, Brown recently observed that the Hodge regulator cannot be an isomorphism when $r_1 + \cdots + r_m > 1$ and $d$ is small. Still, one hopes that these are isomorphisms once $d$ is large enough.
The inclusion $\text{MTM}(\mathbb{Z}) \hookrightarrow \text{MHS}$ induces a surjection $\pi_1(\text{MHS}) \to \pi_1(\text{MTM})$.

**Theorem**

The fundamental group of the category of universal mixed elliptic motives is $\pi_1(\text{MTM}) \rtimes \mathcal{G}^{\text{eis}}$ where $\mathcal{G}^{\text{eis}}$ is the maximal quotient of $\mathcal{G}^{\text{rel}}$ on which $\pi_1(\text{MTM})$ acts.

It is called the *Eisenstein quotient* of $\mathcal{G}^{\text{rel}}$. Its DR realization is canonically split: $\mathcal{G}^{\text{eis}}_{\text{DR}} \cong \text{SL}_2 \rtimes \mathcal{U}^{\text{eis}}_{\text{DR}}$. The coordinate ring $\mathcal{O}(\mathcal{U}^{\text{eis}})$ is contained in (and should equal) the set of linear combinations of regularized iterated integrals of Eisenstein series whose periods are MZVs.
Representations of $\pi_1(M_{\mathbb{T}}(\mathbb{Z}) \rtimes G^{\text{eis}})$ are universal mixed elliptic motives. They correspond to “motivic local systems” $\nabla$ over $\mathcal{M}_{1,1}$ whose weight graded quotients are direct sums of $S^m \mathbb{H}(d)$ and whose fiber over $\partial/\partial q$ is in $M_{\mathbb{T}}(\mathbb{Z})$.

Examples of universal mixed elliptic motives

1. the elliptic polylogs of Beilinson–Levin
2. the local system over $\mathcal{M}_{1,2}$ whose fiber over $(E, \text{id}_E, x)$ is the Lie algebra of $\pi_1^{\text{un}}(E', x)$. Its $\mathbb{Q}$ de Rham realization is the elliptic KZB connection studied by Calaque–Etingof–Enriquez and Levin–Racinet.
Pollack Relations

1. The Lie algebra $u^{\text{eis}}$ is not free. This reflects the fact (due to Brown) that the periods of iterated integrals of Eisenstein series contain periods of cusp forms.

2. We know that each normalized eigenform gives a countable set of minimal relations (up to the actions of $\text{SL}_2$ and $G_\mathbb{Q}$) between the Eisenstein generators.

3. Standard conjectures about the size of $\text{Ext}^1_{\text{mot}}(\mathbb{Q}, V_f(d))$ imply that there are no more relations.
Set $\mathfrak{p} = \pi_1^{un}(E_{\partial/\partial q}, \vec{w})$. It is isomorphic to the rank 2 free Lie algebra $\mathbb{L}(H)^\wedge$. The monodromy action induces a homomorphism

$$\mathfrak{k} \rtimes \mathfrak{g}_{\text{eis}} \to \text{Der} \, \mathbb{L}(H)^\wedge$$

where $\mathfrak{k} = \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots)^\wedge$.

The generator $e_{2n}$ of $u^{\text{eis}}$ corresponding to $G_{2n}$ goes to a derivation $\epsilon_{2n}$ of $\mathbb{L}(H)$; the generator $\sigma_{2m-1}$ determines a derivation $\zeta_{2m-1}$ that corresponds to $G_{2m}$.

Relations between the generators of $\mathfrak{k} \rtimes \mathfrak{g}_{\text{eis}}$ give relations between the $\epsilon_{2n}$'s and $\zeta_{2m-1}$'s.
Pollack (2009) found such relations in degree 2, and (mod “depth 3”) found relations of all degrees. All of his relations are known to lift to $u^{eis}$ and they account for all known relations.

**Example: quadratic relations**

$$\sum_{a+b=n} c_a [\epsilon_{2a+2}, \epsilon_{2b+2}] = 0$$

if and only if there is a cusp form $f$ of $SL_2(\mathbb{Z})$ of weight $2n + 2$

with $r_f^+(a, b) = \sum c_a a^{2a} b^{2n-2a}$

He also found remarkable congruences between several of these relations. One example is given on the following slides.
The “quartic” relation corresponding to the normalized cusp form of weight 12 is a multiple of

\[ 0 = 19958400 \omega_{12,2}^4 - 383090400 \omega_{10,4}^4 + 1149271200 \omega_{8,6}^4 \]
\[ - 1134826056 [\epsilon_4, \omega_{8,2}^3] + 94270176 [\epsilon_4, \omega_{6,4}^3] \]
\[ + 3552068520 [\epsilon_6, \omega_{6,2}^3] - 691691000 [\epsilon_6, \omega_{4,4}^3] \]
\[ - 2708723160 [\epsilon_8, \omega_{4,2}^3] + 674053380 [\epsilon_{10}, \omega_{2,2}^3] \]
\[ + 808632825 [\epsilon_4, [\epsilon_4, [\epsilon_6, \epsilon_4]]] \]
This is congruent mod 691 to 318 times the relation corresponding to $691G_{12}$:

$$691[\zeta_3, \epsilon_{12}] = -210\omega_{12,2}^4 + \frac{30979}{165} [\epsilon_4, \omega_{8,2}^3] - \frac{468}{5} [\epsilon_4, \omega_{6,4}^3] + 429 [\epsilon_6, \omega_{6,2}^3] + 429 [\epsilon_8, \omega_{4,2}^3] + \frac{1742}{3} [\epsilon_{10}, \omega_{2,2}^3] - \frac{819}{8} [\epsilon_4, [\epsilon_4, [\epsilon_6, \epsilon_4]]].$$

He computes another in weight 16.

These and Brown’s computations of the periods of twice iterated integral of Eisenstein series suggest that there is an action of the Hecke algebra on (say) the cohomology groups

$$H^\bullet\left(\pi_1(\text{MTM}(\mathbb{Z}) \ltimes \mathcal{G}^{\text{eis}}, S^{2n}H(d))\right) \cong \text{Ext}^\bullet_{\text{MEM}}(\mathbb{Q}, S^{2n}\mathbb{H}(d)),$$

where MEM denotes the category of universal mixed elliptic motives over $\mathcal{M}_{1,1}$. 
Future Projects and Possible Applications (with Brown)

1. Investigate analogues (two kinds) of Beilinson’s conjecture in MMM by studying extensions in $\mathcal{O}(G^{rel})$.

2. The problem of constructing Hecke operators in this non-abelian setting.

3. Pollack relations and their manifestations in higher genus; their relationship to the cohomology of outer automorphism groups of free groups.

4. Mixed modular motives and universal elliptic motives of level $N > 1$; their relationship to $\text{MTM}(\mathcal{O}(\mu_N, 1/N))$ and its missing periods.

5. The Broadhurst–Kreimer conjecture.

6. Prove that Pollack’s congruences generalize to all weights.
References


