Problem 1 (20 points)
Consider a one-dimensional quantum-mechanical simple harmonic oscillator of mass \( m \) and potential energy \( \frac{kx^2}{2} \). The energy levels of this system are \( E_n = \left( n + \frac{1}{2} \right) \hbar \omega \) \((n = 0, 1, 2, \ldots)\) where \( \omega^2 = \frac{k}{m} \). The system is in the state \( |n\rangle \).

(a) Find the uncertainty in the potential energy. (10 points)

(b) Find the value of \( \Delta x \Delta p_x \), the products of uncertainties in the position and momentum. (10 points)

Problem 2 (30 points)
A diatomic molecule may be treated as a rigid rotor with moment of inertia \( I \), rotating in the \( xy \) plane and fixed along the \( z \) axis at the midpoint of the two atoms. Let \( \phi \) be the azimuthal angle (that is, the angle in the \( xy \) plane).

The Hamiltonian for such a system can be represented as

\[
\hat{H} = -\frac{\hbar}{2I} \frac{d^2}{d\phi^2}.
\]

This is in close analogy with linear kinetic energy.

(a) Find the energy eigenvalues and eigenfunctions for this system. (20 points)

(b) At \( t = 0 \) the rotor is in the state \( \psi(\phi,0) = A \sin^2 \phi \). Find \( \psi(\phi,t) \). (10 points)
Problem 3 (20 points)
A system has energy levels \( E_n = \frac{\omega \hbar}{n} \), \( n = 1, 2, 3, \ldots \), and \( \omega \) is some positive constant. We label the corresponding eigenstates \( |n\rangle \).

(a) Suppose we know that \( |\psi(t)\rangle = |\psi(0)\rangle \) for \( t = 12\pi/\omega \). Show that \( |\psi(0)\rangle \) may be written as a linear combination of only finitely many of the \( |n\rangle \) states and identify which values of \( n \) are allowed. (10 points)

(b) The system is initially in the state \( |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \). After a time \( t \) we measure the operator \( \hat{A} = |1\rangle \langle 2| + |2\rangle \langle 1| \). Calculate \( \langle A(t) \rangle \). (10 points)

Problem 4 (40 points)
The hyperfine interaction in Hydrogen is an interaction between the 1s electron of spin \( \frac{1}{2} \) and the proton of spin \( \frac{1}{2} \). The effective Hamiltonian for the interaction is

\[
\hat{H} = \frac{A}{D^2} \hat{S} \cdot \hat{I}
\]

where \( \hat{S} \) and \( \hat{I} \) are the spin operators for the electron and proton, respectively, and \( A > 0 \).

(a) Without the hyperfine interaction, states can be labeled by the quantum numbers \( S = \frac{1}{2}, m_S = \pm \frac{1}{2}, I = \frac{1}{2}, \) and \( m_I = \pm \frac{1}{2} \). With the hyperfine interaction, how would you label the energy states? Find the relationship between the two sets of states. (20 points)

(b) What is the result of the hyperfine interaction? (10 points)

(c) If in addition one turns on a small uniform magnetic field in the \( +z \) direction, what happens to the energies of each of these states? (10 points)

Problem 5 (30 points)
For a spin \( \frac{1}{2} \) particle, we define the projection operator \( \hat{P}_n = |+n\rangle \langle +n| \).

(a) Calculate \( \left[ \hat{P}_x, \hat{P}_z \right] \). Relate your answer to a spin operator. (10 points)

(b) Derive an uncertainty relation based on the commutator from part (a). (10 points)

(c) Show that your uncertainty relation is equivalent to \( \Delta S_x \Delta S_z \geq \frac{\hbar}{2} |\langle S_y \rangle| \). (10 points)
Problem 6 (30 points)
The coherent state of a harmonic oscillator is

\[ |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \]

where \( \alpha \) is some complex number and \(|n\rangle\) is the \( n \)-th energy eigenstate.

(a) Show that \(|\alpha\rangle\) is an eigenstate of the lowering operator \( \hat{a} \) and find its corresponding eigenvalue. (10 points)

(b) Find the expected energy of the coherent state \(|\alpha\rangle\). (5 points)

(c) Find the uncertainty in the energy of \(|\alpha\rangle\). (5 points)

(d) Can you find an eigenstate of the raising operator \( \hat{a}^\dagger \)? If so find it. If not explain why not. (10 points)

Problem 7 (40 points)
A particle of mass \( m \) is in an infinite square well with a \( \delta \) function at the center

\[ V(x) = \begin{cases} \infty & x < -a \\ \frac{\alpha \hbar^2}{2m} \delta(x) & -a < x < a \\ \infty & x > a \end{cases} \]

where \( \alpha > 0 \). This potential has parity symmetry, so its bound state wavefunctions are either even or odd functions.

(a) In each region where \( V(x) \) is constant, write the general form of the wavefunction \( \psi(x) \) with energy \( E > 0 \). (10 points)

(b) Derive the boundary conditions which should be imposed on \( \psi(x) \). (10 points)

Hint: Recall from the homework that for a \( \delta \) function of this form we found

\[ \psi'(0^+) - \psi'(0^-) = \alpha \psi(0). \]

(c) Derive an equation for the bound state energies \( E \) for even wavefunctions. (10 points)

(d) Derive an equation for the bound state energies \( E \) for odd wavefunctions. (10 points)