Assignment 10

171.303 Quantum Mechanics I

Due: December 4, 2018

Problem 1

Consider a particle in an energy eigenstate $|n\rangle$ of the harmonic oscillator Hamiltonian $\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$.

(a) Calculate $\langle x\rangle$ and $\langle p_x\rangle$ for this state.

(b) Calculate Δx and Δp_x. Verify that the Heisenberg uncertainty relation is satisfied. Which energy eigenstate(s) (if any) minimize the uncertainty?

Problem 2

The ground state position-space wavefunction is $\langle x|0\rangle = (\frac{m\omega}{\pi\hbar})^{1/4}e^{-m\omega x^2/2\hbar}$.

(a) Using a Fourier transform, find the momentum-space wavefunction $\langle p|0\rangle$.

(b) Use the momentum-space representation of the raising operator $\hat{a}^\dagger$ to find the momentum-space wavefunction of the first excited state $\langle p|1\rangle$.

Hint: Recall that in momentum space, $\hat{x} \to i\hbar\frac{\partial}{\partial p}$.

Problem 3

The commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ (or, equivalently, $[\hat{x}, \hat{p}_x] = i\hbar$) is only possible for an infinite dimensional space.

(a) Suppose we have $\left[\hat{A}, \hat{B}\right] = \hat{C}$ on a finite-dimensional space. Show that $\operatorname{tr}\hat{C} = 0$. This is incompatible with $\hat{C} = 1$ since the identity operator has nonzero trace.

(b) Now considering the harmonic oscillator, calculate the energy basis matrix representations of $\hat{a}$ and $\hat{a}^\dagger$, and verify the commutator identity $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ in matrix form.

(c) Check that the operators $\hat{a}^\dagger \hat{a}$ and $\hat{a}\hat{a}^\dagger$ do not have well-defined traces. This loophole allows us to get around your results from (a), but only in infinite dimensional spaces.