Problem 1

(a) The given information tells us that \(|\psi\rangle = c_+ |+z\rangle + c_- |-z\rangle\), with \(|c_+|^2 = 2/3\) and \(|c_-|^2 = 1/3\). We can use our global phase freedom to set \(c_+\) to be real and positive (other choices are also possible), \(c_+ = \sqrt{2/3}\). We are left with \(c_- = \frac{1}{\sqrt{3}} e^{i\phi};\) this relative phase is a free (real) parameter and cannot be eliminated. Thus our answer is (either form is acceptable):

\[ |\psi\rangle = \sqrt{\frac{2}{3}} |+z\rangle + \frac{e^{i\phi}}{\sqrt{3}} |-z\rangle \rightarrow \left( \begin{array}{c} \sqrt{\frac{2}{3}} e^{i\phi} \\ \frac{1}{\sqrt{3}} \end{array} \right) \]

(b) We see that \(\langle S_z \rangle = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}\). Since \(\hat{S}_z^2 = \frac{\hbar^2}{4}\) in general for spin-\(\frac{1}{2}\), we have

\[ \Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2} = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{36}} = \frac{\sqrt{2}}{3} \hbar. \]

This doesn’t depend on the relative phase \(\phi\); indeed it could not. \(\Delta S_z\) is a statistical quantity which only depends on the probability distribution for \(S_z\), and that distribution is given to us in the problem. The relative phase \(\phi\) cannot appear in expressions unless they involve observables other than \(S_z\).

(c) We calculate \(\langle S_z \rangle\) as a function of \(\phi\).

\[ \langle S_z \rangle = \langle \psi | \hat{S}_z | \psi \rangle = \left( \begin{array}{c} \sqrt{\frac{2}{3}} e^{-i\phi} \\ \frac{1}{\sqrt{3}} \end{array} \right) \frac{\hbar}{2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} \sqrt{\frac{2}{3}} \\ e^{i\phi} \end{array} \right) \]

\[ = \frac{\hbar}{2} \sqrt{\frac{2}{3}} \left( e^{-i\phi} + e^{i\phi} \right) \]

\[ = \frac{\hbar}{3} \sqrt{2} \cos \phi. \]
We are told that this is maximized among all such states, which is accomplished precisely when \( \cos \phi = 1 \). Thus we have

\[
\langle S_z \rangle = \frac{\sqrt{2}}{3} \hbar,
\]

\[
|\psi\rangle = \sqrt{\frac{2}{3}} |+z\rangle + \frac{1}{\sqrt{3}} |-z\rangle.
\]

Note that the state is completely determined at this point.

(d) The commutator \([\hat{S}_y, \hat{S}_z]\) gives us an uncertainty relation

\[
\Delta S_y \geq \frac{\hbar}{2} |\langle S_x \rangle|.
\]

The numbers we found in parts (b) and (c) give us then \( \Delta S_y \geq \frac{\hbar}{2} \). Note that this bound is very strong; for a spin-\( \frac{1}{2} \) particle, \( \Delta S_y \) is always between 0 and \( \frac{\hbar}{2} \).

(e) We know that \( \hat{S}_y^2 = \frac{\hbar^2}{4} \); we are left needing to calculate \( \langle S_y \rangle \).

\[
\langle S_y \rangle = \langle \psi | \hat{S}_y | \psi \rangle
\]

\[
= \left( \begin{matrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{matrix} \right) \frac{\hbar}{2} \left( \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \right) \left( \begin{matrix} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} \end{matrix} \right)
\]

\[
= 0.
\]

Thus \( \Delta S_y = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2} \), which satisfies the bound.

Problem 2

(a) The Hamiltonian is a multiple of \( \hat{S}_x \), so its eigenvectors are \( |\pm x\rangle = \frac{1}{\sqrt{2}} (|+z\rangle \pm |-z\rangle) \). The corresponding eigenvalues are \( \pm \frac{\hbar}{2} \) (because we know the eigenvalues of \( \hat{S}_x \) are \( \pm \frac{\hbar}{2} \)).

(b) Our initial state is \( |\psi(0)\rangle = |+z\rangle = \frac{1}{\sqrt{2}} (|+x\rangle + |-x\rangle) \). Writing it in this basis, it is easier to apply the time evolution operator:
\[ |\psi(t)\rangle = e^{-i\hat{H}t/\hbar} \frac{1}{\sqrt{2}} (|+x\rangle + |-x\rangle) \]
\[ = \frac{1}{\sqrt{2}} (e^{-i\omega t/2} |+x\rangle + e^{i\omega t/2} |-x\rangle) \]
\[ = \frac{1}{2} \left( (e^{-i\omega t/2} + e^{i\omega t/2}) |+z\rangle + (e^{-i\omega t/2} - e^{i\omega t/2}) |-z\rangle \right) \]
\[ = \cos \frac{\omega t}{2} |+z\rangle - i \sin \frac{\omega t}{2} |-z\rangle. \]

(c) Viewing \( e^{-i\hat{H}t/\hbar} \) as a rotation operator, it is \( \hat{R}(\omega t) \). If we rotate the spin polarization vector \( k \), it should rotate by \( \frac{3\pi}{2} + 2n\pi \) to rotate to \( j \). Thus the first such time should be at \( t_0 = \frac{3\pi}{2\omega} \) (assuming \( \omega \) is positive; if it is negative the first time is instead at \( -\frac{\pi}{2\omega} \)).

We can check this manually. The probability that a measurement after time \( t_0 \) will find the particle in the state \( |+\rangle \) is given as

\[ P(S_y = \hbar/2) = |\langle +y | \psi(t) \rangle|^2 \]
\[ = \left| \frac{1}{\sqrt{2}} (1 - i) \left( \cos \frac{\omega t}{2} \right) \right|^2 \]
\[ = \frac{1}{2} \left( \cos \frac{\omega t}{2} - \sin \frac{\omega t}{2} \right)^2 \]
\[ = \frac{1}{2} (\sqrt{2} \sin \left( \frac{\pi}{4} - \omega t \right))^2 \]
\[ = \sin^2 \left( \frac{\pi}{4} - \omega t \right). \]

It is clear that this does become 1; the first time (assuming \( \omega > 0 \) is when the argument of the \( \sin^2 \) is \( -\frac{\pi}{2} \), giving \( t_0 = \frac{3\pi}{2\omega} \).

(d) Changing the charge to \( +e \) means the Hamiltonian becomes \( -\omega \hat{S}_x \). We can use the same expression from part (b), but with \( \omega \rightarrow -\omega \) (or, interestingly, \( t \rightarrow -t \)). Using the reflection properties of \( \sin \) and \( \cos \) we have

\[ |\psi(t)\rangle = \cos \frac{\omega t}{2} |+z\rangle - i \sin \frac{\omega t}{2} |-z\rangle. \]

Problem 3

(a) The state is \( |1, 0\rangle_x = \frac{1}{\sqrt{2}} (|1, 1\rangle - |1, -1\rangle) \). The rotation operator we are interested in is \( \hat{R}(\pi k) = e^{-i\pi \hat{S}_z/2k} \). We see that this is diagonal in the \( |1, m_z\rangle \) basis, with eigenvalues \( e^{-i\pi/2}, 1, e^{i\pi/2} \) for \( m_z = 1, 0, -1 \), respectively. Applying linearity we then have
(b) As we have rotated the state by $\pi/2$ along the $z$ axis, it is an eigenstate of the operator we get when we rotate the operator $\hat{S}_x$ by the same rotation. It is clear that this is $\hat{S}_y$. One could explicitly prove this by using the matrices for $\hat{R}$, $\hat{S}_x$, and $\hat{S}_y$, and show $\hat{R}\hat{S}_x\hat{R}^\dagger = \hat{S}_y$ (though this was not required).

(c) We have $\hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-)\), so we need to find the matrices for the raising and lowering operators. The raising operator satisfies $\hat{S}_+ |1,m_z\rangle = \sqrt{2} - m_z(m_z + 1)\hbar |1, m_z + 1\rangle$, from which we can read off the matrix as

$$\hat{S}_+ \to \hbar \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

The lowering operator is the conjugate-transpose of this since $\hat{S}_y^\dagger = \hat{S}_-$. Then we have

$$\hat{S}_y \to \frac{1}{2i} \hbar \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$ 

(d) We can simply check by using matrix multiplication.

$$\hat{S}_y \hat{R}(\frac{\pi}{2}k) |1,0\rangle_x \to \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ 

Thus $\hat{S}_y \hat{R}(\frac{\pi}{2}k) |1,0\rangle_x = 0 = 0 \hat{R}(\frac{\pi}{2}k) |1,0\rangle_x$, so $\hat{R}(\frac{\pi}{2}k) |1,0\rangle_x$ is an eigenvector of $\hat{S}_y$ with eigenvalue 0. We could identify this as $|1,0\rangle_y$, but more commonly we would define $|1,0\rangle_y$ by removing the global phase of $-i$ in the definition, which does not change the physical state.