Problem 1 (40 points)

A spin-$\frac{1}{2}$ particle is in a state $|\psi\rangle$ such that if we measure $S_z$, we have a $\frac{2}{3}$ chance of finding $S_z = +\frac{\hbar}{2}$.

(a) Write a general form of $|\psi\rangle$ in the $|\pm z\rangle$ basis, using as few free parameters as possible. (8 points)

(b) What is $\Delta S_z$ for this state? Does it depend on the parameters in part (a)? (7 points)

(c) We further know that, among all such states, $|\psi\rangle$ is chosen to maximize $\langle S_x \rangle$. What is $\langle S_x \rangle$, and what is the general form of $|\psi\rangle$ with this additional information? (10 points)

(d) Your results from parts (b) and (c) can be used to put a bound on $\Delta S_y$. What is this bound? (7 points)

(e) Calculate $\Delta S_y$ and verify the bound is satisfied. (8 points)
Problem 2 (30 points)

A beam of electrons (spin-$\frac{1}{2}$) is initially in the state $|+z\rangle$. Starting at time $t = 0$ it is subjected to a magnetic field $\mathbf{B} = B_0 \mathbf{i}$ (where $\mathbf{i}$ is the unit vectors in the $x$ direction). The corresponding Hamiltonian is $\hat{H} = \omega \hat{S}_x$.

(a) Find the energy levels and states for this Hamiltonian in the $|\pm z\rangle$ basis. (7 points)

(b) Find the time-dependent state of the particle $|\psi(t)\rangle$ in the $|\pm z\rangle$ basis. (10 points)

(c) Is there some time $t_0$ such that, if we measure $S_y$ at time $t_0$, 100% of the particles will have $S_y = +\frac{\hbar}{2}$? If so find the first such time. (8 points)

(d) Positrons are spin-$\frac{1}{2}$ particles with the same properties as electrons except with opposite charge $q = +e$. Find the time-dependent state $|\psi(t)\rangle$ if we instead send a beam of positrons with initial state $|+z\rangle$ through the same magnetic field. (5 points)

Problem 3 (30 points)

The state $|1,0\rangle_x$ is an eigenstate of $\hat{S}_x$ with eigenvalue 0. It is represented in the basis of $|1, m_z\rangle$ vectors as

\[
|1,0\rangle_x \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.
\]

(a) After applying a rotation by $\pi/2$ about the $z$ axis to the state $|1,0\rangle_x$, what is the resulting state? (10 points)

(b) The state you found in part (a) is an eigenvector of a spin operator. Which operator? (5 points)

(c) Find the matrix representing the operator in part (b). (10 points)

(d) Verify that the vector you found in part (a) is indeed an eigenvector of the matrix from part (c). (5 points)