Assignment 9

171.303 Quantum Mechanics I

Due: November 27, 2017

Problem 1

Consider a particle of mass m in the ground state of an infinite potential well

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{else.} \end{cases}$$

At time $t = 0$ the right barrier is moved to the position $x = 2L$. This happens suddenly so that the wavefunction of the state doesn't change instantaneously.

(a) Suppose we immediately measure the energy of the state in the new potential well. What are the lowest two possible energies that we could measure? Find the probability of each.

(b) Do we have a nonzero probability for measuring every energy level of the new well? If so, explain why. If not, what is the lowest energy level which we would never find? You should be able to explain your reasoning without any complicated calculations.

(c) As the state evolves in time with its new Hamiltonian, which of the following quantities change? (You should not need any detailed calculations to solve this.)

(i): $\langle x \rangle$

(ii): $\langle p_x \rangle$

(iii): The probability that the particle has position in the range $x_1 \leq x \leq x_2$

(iv): The probability that the particle has momentum in the range $p_1 \leq p_x \leq p_2$

Problem 2

One example of a one-dimensional system which can be solved exactly is the delta-well potential

$$V(x) = -\alpha \frac{\hbar^2}{2m} \delta(x)$$

where α is a positive constant with dimensions of inverse length. It can be thought of as a limiting case of a finite potential well which has a small width but such that the area above the well is finite.

(a) Since the delta-well is infinite at $x = 0$, the wavefunction $\psi(x) = \langle x|\psi\rangle$ doesn't need to be differentiable at $x = 0$. Show that for an energy eigenstate the discontinuity in the derivative is given by

$$\frac{\partial \psi}{\partial x}|_{0^+} - \frac{\partial \psi}{\partial x}|_{0^-} = -\alpha\psi(0)$$

by integrating the time-independent Schrödinger equation.

(b) In the two regions $x < 0$, $x > 0$, the potential is 0. Solve for the energy level(s) and bound state wavefunction(s) by finding the solutions in these regions and imposing the constraint from part (a). Normalize your wavefunction(s).

(c) For the bound state(s) you found in part (b), determine the momentum space wavefunction $\phi(p) = \langle p|\psi\rangle$.

Problem 3

Consider scattering off a delta-function potential barrier:

$$V(x) = \alpha\frac{\hbar^2}{2m}\delta(x)$$

(a) Calculate the transmission and reflection coefficients for an incident particle of mass m tunneling through the barrier by solving the Schrödinger equation using the boundary conditions from problem 2.

(b) Derive the same results by considering the delta function as a limiting case of a finite square well barrier (see equation (6.144)).

Problem 4

We have a particle of mass m confined in a potential well which is infinite on one side and finite on the other, given by

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < L \\ V_0 & x > L \end{cases}$$

for V_0 some positive constant with dimensions of energy. This potential could be used as a very rough model for the van der Waals force experienced by a molecule in the presence of another molecule.

(a) By applying the various boundary conditions, show that the bound state energy levels satisfy the equation[1]

$$\tan\sqrt{\frac{2mL^2E}{\hbar^2}} = -\sqrt{\frac{E}{V_0 - E}}.$$

Note: While $E = 0$ solves this equation, it is easy to check that there is no normalizable wavefunction which is an eigenfunction of this Hamiltonian with $E = 0$. The solution is spurious, and arises from our method for solving the differential equation which is only valid when $E > 0$.

(b) Sketch a graph (accurate enough to show the key features relevant to your analysis) of the left-hand side of the equation as a function of $\sqrt{E}$, and the right hand side as a function of $\sqrt{E}$ for at least two different values of V_0. Identify the points corresponding to the bound-state energies. Are there infinitely many or finitely many bound states? Is there a value of V_0 for which there are no bound states?

[1] This is a transcendental equation, which can not be solved exactly in terms of simple functions. In most cases one resorts to solving it numerically, looking at asymptotics, or doing qualitative analysis.