Assignment 8

171.303 Quantum Mechanics I

Due: November 13, 2018

Problem 1

Let P be a positive constant with units of momentum. Consider the wave packet defined by

$$\langle p|\psi\rangle = \begin{cases} N & |p| \leq P \\ 0 & |p| > P \end{cases}$$

(a) Determine the value of N such that $|\psi\rangle$ is normalized.

(b) Find the position space wavefunction $\langle x|\psi\rangle = \psi(x)$.

(c) Estimate (you do not need to explicitly calculate) Δx and Δp_x, and show that the product $\Delta x \Delta p_x$ satisfies the Heisenberg uncertainty principle.

Problem 2

The initial state of a free particle of mass m confined to a line is described by the Gaussian wavefunction

$$\psi(x,0) = \langle x|\psi(0)\rangle = (\pi a^2)^{-1/4} e^{-x^2/2a^2}$$

(a) Show that the time-dependent wave function is given by

$$\psi(x,t) = \langle x|\psi(t)\rangle = \left(\pi\left(a + \frac{i\hbar t}{ma}\right)^2\right)^{-1/4} \exp\left(-\frac{x^2}{2(a^2 + i\hbar t/m)}\right)$$

(b) Calculate $\Delta x(t)$ and $\Delta p_x(t)$. Check the uncertainty relation holds for all times. At what times do we have minimum uncertainty?
Hint: Some of the steps in this problem may be easier in momentum-space.

Problem 3

Throughout this problem, for any operator $\hat{A}$, let $\langle A \rangle_i$ be the expectation of $\hat{A}$ in the state $|\psi_i\rangle$ (to be defined). Let $|\psi_0\rangle$ be any state in a Hilbert space for a particle confined on a line without any additional internal degrees of freedom (i.e. spin-0).

(a) Let $|\psi_1\rangle = \hat{T}(\Delta x)|\psi_0\rangle$ for some translation $\hat{T}(\Delta x)$. Show that

$$\langle x \rangle_1 = \langle x \rangle_0 + \Delta x \qquad \text{and} \qquad \langle p_x \rangle_1 = \langle p_x \rangle_0.$$

(b) Let $|\psi_2\rangle = e^{ip_0\hat{x}/\hbar}|\psi_0\rangle$. Calculate $\langle x \rangle_2$ and $\langle p_x \rangle_2$ in terms of $\langle x \rangle_0$, $\langle p_x \rangle_0$, and p_0.

(c) Let $|\psi_3\rangle$ be a state with a *real* position-basis wave function $\psi_3(x) = \psi_3^*(x) = \langle x|\psi_3\rangle$. Show that $\langle p_x \rangle_3 = 0$.

Problem 4

A particle in the potential well

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}$$

is in the state

$$\psi(x) = \begin{cases} Nx(x-L) & 0 < x < L \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Determine the (real, positive) value of N such that the state is normalized.

(b) What is the probability that a measurement of energy yeilds the ground state of the well? What about the first excited state? Calculate the answers exactly and then find the numerical value.

(c) What is $\langle E \rangle$ for this state? Calculate the ratio $\langle E \rangle / E_1$ (where E_1 is the ground state energy) exactly and find the numerical value.