Chapter 4 Problems

171.303 Quantum Mechanics I

These problems will not be graded, but you are encouraged to work on them before the midterm.

Problem 1

A photon propagating (which is a 2-state system) is propagating along the z-axis in a crystal.[1] In the linear polarization basis $|X\rangle, |Y\rangle$, the Hamiltonian is given by the matrix

$$\hat{H} = \frac{E_0}{\hbar}\hat{S}_z \to \begin{pmatrix} 0 & -iE_0 \\ iE_0 & 0 \end{pmatrix}.$$

(a) What are the energy levels? Find a state corresponding to each energy level.

(b) The photon initially enters in the state $|\psi(0)\rangle = |Y\rangle$. Find $|\psi(t)\rangle$. Is the photon still linearly polarized after time t? If so, in what direction?

Note: A photon propagating along the z axis is linearly polarized if it has *real* components in the linear polarization basis. The direction of polarization is $\mathbf{x}' = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$ if its state is $|X'\rangle = \cos\phi|X\rangle + \sin\phi|Y\rangle$.

Problem 2

A spin-1 particle with magnetic moment $\boldsymbol{\mu} = \frac{gq}{2mc}\mathbf{S}$ is situated in a constant magnetic field $\mathbf{B} = B_0\mathbf{k}$ in the z-direction. At time $t = 0$, the particle is in the state $S_x = -\hbar$.

(a) Find the state of the particle $|\psi(t)\rangle$ as a function of time.

(b) Calculate $\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle$ as functions of time.

(c) Calculate $\Delta S_x, \Delta S_z$ as functions of time. Show that the uncertainty relation $\Delta S_x \Delta S_z \geq \frac{\hbar}{2}|\langle S_y\rangle|$ is satisfied for all times. Are the two sides ever equal?

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[1] Photons are spin-1 systems with only 2 states, $S_z = \pm\hbar$. Such a particle is not possible in the framework we studied in chapter 3; understanding this requires applying special relativity. For now you may treat it as any other 2-state system.

Problem 3

A spin-$\frac{1}{2}$ particle is placed in a region with constant magnetic field in the z direction and a rotating field in the x and y directions, given by

$$\mathbf{B} = B_0\mathbf{k} + B_1(\sin(\omega t)\mathbf{i} - \cos(\omega t)\mathbf{j}).$$

The Hamiltonian can be written as

$$\hat{H}(t) = \omega_0\hat{S}_z + \omega_1(\sin(\omega t)\hat{S}_x - \cos(\omega t)\hat{S}_y).$$

The particle is initially in the state $|\psi(0)\rangle = |-\mathbf{z}\rangle$. Find the probability $p_+(t)$ as a function of time that $|\psi(t)\rangle$ is in the state $|+\mathbf{z}\rangle$. What is the maximum probability over all times, and what is the first time at which $p_+(t)$ reaches this maximum value?

Hint: You may find section 4.4 helpful.

Problem 4

A spin-$\frac{3}{2}$ particle is initially in the state $|\psi(0)\rangle = |\frac{3}{2}, \frac{3}{2}\rangle$. It is subjected to the Hamiltonian $\hat{H} = \omega\hat{S}_x$.

(a) Suppose only a small amount of time Δt passes before we measure S_z for this particle. Calculate the probability that we measure the state $|\frac{3}{2}, -\frac{3}{2}\rangle$ to the lowest order in Δt at which the probability is nonzero.

(b) Now we perform the same experiment but wait for an amount of time $t = \pi/\omega$ before doing the measurement. What is the (exact) probability of finding the state $|\frac{3}{2}, -\frac{3}{2}\rangle$ now? Explain.