Chapter 4 Problems

171.303 Quantum Mechanics I

These problems will not be graded, but you are encouraged to work on them before the midterm.

Problem 1

A photon propagating (which is a 2-state system) is propagating along the $z$-axis in a crystal. In the linear polarization basis $|X\rangle, |Y\rangle$, the Hamiltonian is given by the matrix

$$\hat{H} = \frac{E_0}{\hbar} \hat{S}_z \rightarrow \begin{pmatrix} 0 & -iE_0 \\ iE_0 & 0 \end{pmatrix}.$$  

(a) What are the energy levels? Find a state corresponding to each energy level.

(b) The photon initially enters in the state $|\psi(0)\rangle = |Y\rangle$. Find $|\psi(t)\rangle$. Is the photon still linearly polarized after time $t$? If so, in what direction?

Note: A photon propagating along the $z$ axis is linearly polarized if it has real components in the linear polarization basis. The direction of polarization is $x' = \cos \phi i + \sin \phi j$ if its state is $|X'\rangle = \cos \phi |X\rangle + \sin \phi |Y\rangle$.

Problem 2

A spin-1 particle with magnetic moment $\mu = \frac{gq}{2mc} S$ is situated in a constant magnetic field $B = B_0 k$ in the $z$-direction. At time $t = 0$, the particle is in the state $S_x = -\hbar$.

(a) Find the state of the particle $|\psi(t)\rangle$ as a function of time.

(b) Calculate $\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle$ as functions of time.

(c) Calculate $\Delta S_x, \Delta S_z$ as functions of time. Show that the uncertainty relation $\Delta S_x \Delta S_z \geq \frac{\hbar}{2} |\langle S_y \rangle|$ is satisfied for all times. Are the two sides ever equal?

1Photons are spin-1 systems with only 2 states, $S_z = \pm \hbar$. Such a particle is not possible in the framework we studied in chapter 3; understanding this requires applying special relativity. For now you may treat it as any other 2-state system.
Problem 3

A spin-$\frac{1}{2}$ particle is placed in a region with constant magnetic field in the $z$ direction and a rotating field in the $x$ and $y$ directions, given by

$$\mathbf{B} = B_0 \hat{\mathbf{k}} + B_1 (\sin(\omega t) \hat{\mathbf{i}} - \cos(\omega t) \hat{\mathbf{j}}).$$

The Hamiltonian can be written as

$$\hat{H}(t) = \omega_0 \hat{S}_z + \omega_1 (\sin(\omega t) \hat{S}_x - \cos(\omega t) \hat{S}_y).$$

The particle is initially in the state $|\psi(0)\rangle = |-z\rangle$. Find the probability $p_+(t)$ as a function of time that $|\psi(t)\rangle$ is in the state $|+z\rangle$. What is the maximum probability over all times, and what is the first time at which $p_+(t)$ reaches this maximum value?

*Hint:* You may find section 4.4 helpful.

Problem 4

A spin-$\frac{3}{2}$ particle is initially in the state $|\psi(0)\rangle = |\frac{3}{2}, \frac{3}{2}\rangle$. It is subjected to the Hamiltonian $\hat{H} = \omega \hat{S}_x$.

(a) Suppose only a small amount of time $\Delta t$ passes before we measure $S_z$ for this particle. Calculate the probability that we measure the state $|\frac{3}{2}, -\frac{3}{2}\rangle$ to the lowest order in $\Delta t$ at which the probability is nonzero.

(b) Now we perform the same experiment but wait for an amount of time $t = \pi/\omega$ before doing the measurement. What is the (exact) probability of finding the state $|\frac{3}{2}, -\frac{3}{2}\rangle$ now? Explain.