171.303 Quantum Mechanics Final Exam
Solutions

December 21, 2017 9:00 - 12:00

Problem 1
(a) We are given the $S_x$ basis representation, which tells us that the state is

$$|\psi\rangle = \frac{1}{\sqrt{5}}(|+x\rangle + 2i|-x\rangle) = \frac{1}{\sqrt{10}}((1 + 2i)|+z\rangle + (1 - 2i)|-z\rangle).$$

The possible measurable values for $S_z$ are $\pm \hbar/2$. The probability for each is

$$P(S_z = +\hbar/2) = |\langle +z|\psi\rangle|^2 = \frac{1 + 2i}{\sqrt{10}} = \frac{1}{2},$$

$$P(S_z = -\hbar/2) = |\langle -z|\psi\rangle|^2 = \frac{1 - 2i}{\sqrt{10}} = \frac{1}{2}. $$

(b) The possible values we can find for a measurement of $S_y$ are $\pm \hbar/2$. The probabilities are

$$P(S_y = +\hbar/2) = |\langle +y|\psi\rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}}((+z|\psi\rangle - i(-z|\psi\rangle))^2 \right|$$

$$= \left| \frac{(1 + 2i) - i(1 - 2i)}{\sqrt{20}} \right|^2$$

$$= \left| \frac{-1 - i}{\sqrt{20}} \right|^2$$

$$= \frac{2}{20} = \frac{1}{10}.$$
\[ P(S_y = -\hbar/2) = |\langle -y|\psi \rangle|^2 \]
\[ = \frac{1}{\sqrt{2}} ((+z|\psi \rangle + i(-z|\psi \rangle))^2 \]
\[ = \frac{(1 + 2i) + i(1 - 2i)}{\sqrt{20}} \]
\[ = \frac{3 + 3i}{\sqrt{20}} \]
\[ = \frac{18}{20} = \frac{9}{10}. \]

**Problem 2**

(a) For the state to be normalized, we need

\[ 1 = \langle \psi|\psi \rangle = \int dx \langle \psi|x\rangle \langle x|\psi \rangle \]
\[ = |N|^2 \int_{-\infty}^{\infty} -\infty \sech^2(ax) dx \]
\[ = |N|^2 \left[ \frac{1}{a} \tanh(ax) \right]_{-\infty}^{\infty} \]
\[ = \frac{|N|^2}{a} (1 - (-1)) \]
\[ = \frac{2}{a} |N|^2. \]

The real, positive solution for \( N \) to this equation is \( N = \sqrt{a/2} \).

(b) We were not told the energy of the state so we will solve first for \( V(x) - E \). The time-independent Schrödinger equation in position space is

\[ \frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} = (V(x) - E)\psi(x). \]

We must calculate \( \frac{d^2 \psi(x)}{dx^2} \). The result is

\[ \frac{d\psi(x)}{dx} = -Na \sech(ax) \tanh(ax) \]
\[ \frac{d^2 \psi(x)}{dx^2} = Na^2 (\sech(ax) \tanh^2(ax) - \sech^3(ax)) \]
\[ = Na^2 \sech(ax) (\tanh^2(ax) - \sech^2(ax)) \]
\[ = Na^2 \sech(ax) (1 - 2 \sech^2(ax)). \]
By dividing both sides of the Schrödinger equation by $\psi(x)$ (which is never 0) we see that we have

$$V(x) - E = \frac{\hbar^2}{2m} a^2 (1 - 2 \text{sech}^2(ax)).$$

So picking $E$ appropriately, we can eliminate the first (constant) term and we find a potential which works

$$V(x) = -\frac{\hbar^2 a^2}{m} \text{sech}^2(ax).$$

Other potentials are possible by choosing different energies $E$ but they are all related by just a constant shift and so describe the same physical scenario.

**Problem 3**

(a) We have four regions. In the regions $x < -a$ and $x > a$ the potential is infinite; consequently the wavefunction must be identically 0. In the region $-a < x < 0$ we have a constant potential and $E > V$ (where $V = 0$) so the wavefunction can be written as a linear combination of $\sin(kx)$ and $\cos(kx)$ where $k = \sqrt{2mE}/\hbar$. In the region $0 < x < a$ we again have a constant potential but $E < V$ (where $V = V_0$) so the wavefunction is a linear combination of $e^{qx}$ and $e^{-qx}$ for $q = \sqrt{2m(V_0 - E)}/\hbar$.

Thus our general solution can be written in terms of four unknown constants $A, B, C, D$ as

$$\psi(x) = \begin{cases} 
0 & x < -a \\
A \sin(kx) + B \cos(kx) & -a < x < 0 \\
Ce^{qx} + De^{-qx} & 0 < x < a \\
0 & a < x 
\end{cases}$$

(b) We will have four boundary conditions: one from $x = -a$, one from $x = +a$, and two from $x = 0$. We write them down in abstract form here, without applying them to this particular wavefunction yet. Let $\psi_\pm$ be the wavefunction in the region between 0 and $\pm a$.

$$\psi_-(a) = 0 \\
\psi_-(0) = \psi_+(0) \\
\psi'_-(0) = \psi'_+(0) \\
\psi_+(+a) = 0$$

(c) Continuity of $\psi(x)$ at $x = -a$ requires $\psi(-a) = 0$. We could solve this in terms of $A$ and $B$ but it will be slightly easier to change variables. Let $\psi(x) = A' \sin(k(x + a)) + B' \cos(k(x + a))$ in the region $-a < x < 0$. We could compute $A'$ and $B'$ in terms of $A$ and $B$ by the angle addition formula but it
will not be important for anything so we will omit it. Continuity at \( x = -a \) requires \( B' = 0 \).

Continuity of \( \psi(x) \) at \( x = a \) requires \( \psi(a) = 0 \). Hence \( C = D e^{-2qa} \). We rewrite the wavefunction in the region \(-a < x < a\) with these two conditions applied:

\[
\psi(x) = \begin{cases} 
A' \sin(k(x + a)) & -a < x < 0 \\
D(e^{-q(x - 2a)} - e^{q(x - 2a)}) & 0 < x < a
\end{cases}
\]

Applying continuity of \( \psi \) at \( x = 0 \) gives us the equation

\[ A' \sin(ka) = D(1 - e^{-2qa}). \]

We could use this to eliminate one of the two but it will be easiest to keep it in this form. Finally, continuity of \( \psi' \) at \( x = 0 \) gives a similar equation

\[ A' k \cos(ka) = Dq(1 + e^{-2qa}). \]

Dividing the second equation by the first gives us a pleasant-looking equation in which \( A' \) and \( D \) are both eliminated

\[ k \cot(ka) = q \coth(qa). \]

The equation can be written in terms of \( E \) using the definitions of \( k \) and \( q \).

**Problem 4**

The ground state is an eigenstate of the first two terms of the Hamiltonian with eigenvalue \( \frac{1}{2} \hbar \omega \). To calculate \( \langle E \rangle \) we need to find \( \langle \lambda x^4 \rangle \). We write \( \hat{x}^4 \) in terms of raising and lowering operators:

\[
\hat{x}^4 = \frac{\hbar^2}{4m^2 \omega^2} (\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a} \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} \hat{a} \hat{a}^\dagger + \cdots)
\]

where the \( \cdots \) indicates terms with an unequal number of raising and lowering operators (10 such terms in all). These latter terms do not contribute to the expectation value in any energy eigenstate. In the ground state \( |0\rangle \) we see that to have nonzero expectation we also need terms which start with \( \hat{a} \) and end with \( \hat{a}^\dagger \) to get a nonzero result, which eliminates the first 4 of the 6. The remaining equation can be written as

\[
\langle \lambda x^4 \rangle = \frac{\lambda \hbar^2}{4m^2 \omega^2} (\langle 0| \hat{a} \hat{a} \hat{a}^\dagger \hat{a}^\dagger |0\rangle + \langle 0| \hat{a} \hat{a} \hat{a}^\dagger \hat{a}^\dagger |0\rangle)
\]

Applying \( \hat{a} |n\rangle = \sqrt{n} |n - 1\rangle \) and \( \hat{a}^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle \) four times for each of the two terms, we see that the first term in parentheses is 1 and the second term is 2. Hence the overall result is

\[
\langle E \rangle = \frac{1}{2} \hbar \omega + \frac{3 \lambda \hbar^2}{4m^2 \omega^2}.
\]
Problem 5

(a) We note that $\hat{S}_1 \cdot \hat{S}_2 = \frac{1}{2} \hat{S}_1 z \hat{S}_2 z + \frac{1}{2} \hat{S}_1 - \hat{S}_2 + \frac{1}{2} \hat{S}_1 + \hat{S}_2$. If we take basis vectors $|\frac{1}{2} \frac{1}{2} \rangle, |\frac{1}{2} - \frac{1}{2} \rangle, | - \frac{1}{2} \frac{1}{2} \rangle, | - \frac{1}{2} - \frac{1}{2} \rangle$ then the matrix representing the Hamiltonian is

\[
\hat{H} \rightarrow \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

We see that $|\frac{1}{2} \frac{1}{2} \rangle$ and $| - \frac{1}{2} - \frac{1}{2} \rangle$ are eigenvectors of the matrix immediately. The others come from diagonalizing the central $2 \times 2$ block. We find that the eigenvalues of the $2 \times 2$ matrix (without the prefactor) are $1$ and $-3$. The corresponding eigenvectors are $\frac{1}{\sqrt{2}} (|\frac{1}{2} - \frac{1}{2} \rangle + | - \frac{1}{2} \frac{1}{2} \rangle)$ and $\frac{1}{\sqrt{2}} (|\frac{1}{2} - \frac{1}{2} \rangle - | - \frac{1}{2} \frac{1}{2} \rangle)$, respectively. The four eigenstates are thus

$|I\rangle = |\frac{1}{2} \frac{1}{2} \rangle$

$|II\rangle = \frac{1}{\sqrt{2}} (|\frac{1}{2} - \frac{1}{2} \rangle + | - \frac{1}{2} \frac{1}{2} \rangle)$

$|III\rangle = | - \frac{1}{2} - \frac{1}{2} \rangle$

$|IV\rangle = \frac{1}{\sqrt{2}} (|\frac{1}{2} - \frac{1}{2} \rangle - | - \frac{1}{2} \frac{1}{2} \rangle)$.

(b) The energy levels also follow from our work in part (a). $|I\rangle$, $|II\rangle$, and $|III\rangle$ are triply degenerate, with energy level $\frac{\hbar^2}{4}$. The other state $|IV\rangle$ has energy level $-\frac{3\hbar^2}{4}$.

(c) We note that $\hat{S}_z = \hat{S}_1 z + \hat{S}_2 z + \hat{S}_1 \cdot \hat{S}_2$. Since $\hat{S}_1 z$ and $\hat{S}_2 z$ are just multiples of the identity operator, the eigenstates of $\hat{S}_z$ and $\hat{S}_1 \cdot \hat{S}_2$ agree. Hence our states can be labeled by $s$, where $s = 1$ for $|I\rangle$, $|II\rangle$, and $|III\rangle$, and $s = 0$ for $|IV\rangle$. We need an additional quantum number to describe the degeneracy between the $s = 1$ states, for which we can use $m$, where $m\hbar$ is the eigenvalue of $\hat{S}_z$ (so $m = 1, 0$, and $-1$ for $|I\rangle$, $|II\rangle$, and $|III\rangle$, respectively, and obviously $m = 0$ for $|IV\rangle$). So our total angular momentum kets $|s, m\rangle$ will work to describe the system with this Hamiltonian.

Problem 6

Note that there is a typo in the statement of this problem, which was corrected during the exam. The exponential piece of the wavefunction should be $e^{-\frac{\pi m \omega x^2}{\hbar}}$ rather than $e^{-\frac{\pi \omega x^2}{\hbar}}$. 

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While this problem can be done with the wavefunctions directly, it is easier to use operator methods. As such we need to derive the wavefunctions for the ground state and first excited state of the harmonic oscillator. In fact they are (almost) given on the formula sheet but it is not hard to derive them. The ground state wavefunction satisfies

\[ 0 = \langle x | \hat{a} | 0 \rangle = \sqrt{\frac{m \omega}{2\hbar}} (x \langle x | 0 \rangle + \frac{i \hbar}{m \omega} \frac{d}{dx} \langle x | 0 \rangle). \]

We can see immediately that \( N e^{-\frac{m \omega}{\pi} x^2} \) satisfies this, and we normalize the wavefunction.

\[ 1 = \langle 0 | 0 \rangle = \int_{-\infty}^{\infty} |N|^2 e^{-\frac{m \omega}{\pi} x^2} = |N|^2 \sqrt{\pi \hbar / m \omega} \]

\[ N = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4}. \]

To get the \(|1\rangle\) wavefunction we apply the raising operator. The condition that \( \hat{a}|0\rangle = 0 \) tells us that \( \hat{x}|0\rangle = -\frac{i \hbar}{m \omega} \hat{p}_x |0\rangle \) so we don’t need to use any derivatives here.

\[ \langle x | 1 \rangle = \langle x | \hat{a}^\dagger | 0 \rangle = \sqrt{\frac{2m \omega}{\hbar}} x \langle x | | \rangle 0 \]

\[ = \sqrt{\frac{2m \omega}{\hbar}} x \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m \omega}{\pi} x^2} \]

If we match this onto the given form of \( \psi(x, 0) \), we see that

\[ |\psi\rangle = A \left( \frac{m \omega}{\pi \hbar} \right)^{-1/4} (|0\rangle - |1\rangle). \]

We can see immediately that for the right hand side to be normalized, we need it to be equal to \( \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \), so \( A = \left( \frac{m \omega}{4 \pi \hbar} \right)^{1/4}. \)

(b) This state is a superposition of the \( n = 0 \) and \( n = 1 \) states, each with probability \( \frac{1}{2} \) of being measured. Hence the expectation value of energy is the average of their two energies, which is

\[ \langle E \rangle = \hbar \omega \]

(c) The time-evolved state will be

\[ |\psi(T)\rangle = \frac{1}{\sqrt{2}} e^{-i \omega T/2} (|0\rangle - e^{-i \omega T} |1\rangle). \]

We need \( e^{-i \omega T} = -1 \), and the smallest such value is \( T = \pi / \omega. \)
Problem 7

(a) We have 4 spins to add, and we can add them however we like because addition is associative. Let \( \hat{S}_{12} = \hat{S}_1 + \hat{S}_2 \) and \( \hat{S}_{34} = \hat{S}_3 + \hat{S}_4 \). Then \( \hat{S}_{12} \) and \( \hat{S}_{34} \) have corresponding total angular momentum quantum numbers \( s_{12} \) and \( s_{34} \). Each of them is an addition of two spin-\( \frac{1}{2} \) particles, so we have \( s_{12} = 1, 0 \) and \( s_{34} = 1, 0 \).

Now we need to solve the problem for \( \hat{S} = \hat{S}_{12} + \hat{S}_{34} \). There are four possible combinations to consider, each of which just comes down to applying the fact that the sum of a spin-\( s_a \) and spin-\( s_b \) particle has total angular momentum between \( s_a + s_b \) and \( |s_a - s_b| \) (counting by integer steps).

\[
\begin{align*}
  s_{12} = 1 & \quad s_{34} = 1 & \implies s = 2, 1, 0 \\
  s_{12} = 1 & \quad s_{34} = 0 & \implies s = 1 \\
  s_{12} = 0 & \quad s_{34} = 1 & \implies s = 1 \\
  s_{12} = 0 & \quad s_{34} = 0 & \implies s = 0
\end{align*}
\]

So overall, we have \( s = 2, 1, \) or \( 0 \), corresponding to \( S^2 = 6\hbar^2, 2\hbar^2, 0 \), respectively.

(b) When we say a random state, we mean that all possible states have the same probability. Hence the probability of finding a given value of \( S^2 \) is the number of eigenstates of \( \hat{S}^2 \) with that eigenvalue divided by the total number of states. We just need to count how many \( s = 2, s = 1, \) and \( s = 0 \) states there are.

We see that there is 1 way to form \( s = 2 \), 3 ways to form \( s = 1 \), and 2 ways to form \( s = 0 \). For \( s = 2 \), we have 5 values of \( m \), for 5 total states. For \( s = 1 \) we have 3 values of \( m \) corresponding to 9 total states. For \( s = 0 \) we have only a single value of \( m \) meaning 2 states. We see these sum to 16 as expected since each individual particle has 2 states and \( 2^4 = 16 \).

The probabilities of each possible measurement are then

\[
\begin{align*}
  P(S^2 = 6\hbar^2) &= 5/16 \\
  P(S^2 = 2\hbar^2) &= 9/16 \\
  P(S^2 = 0) &= 1/8
\end{align*}
\]