Assignment 6

171.303 Quantum Mechanics I

Due: October 30, 2018

Problem 1

Two spin-$\frac{1}{2}$ particles with different properties are in a magnetic field of magnitude B. Including their spin-spin interaction, the Hamiltonian is

$$\hat{H} = \omega_1 \hat{S}_{1,z} + \omega_2 \hat{S}_{2,z} + \frac{2A}{\hbar^2} \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2.$$

(a) What are the energies of this system?

(b) Consider the limit $\frac{A}{\hbar} \ll |\omega_1 - \omega_2|, |\omega_1 + \omega_2|$. What are the energy levels to first nonvanishing order in A? Show that these match the results for a pair of noninteracting particles.

(c) Consider the limit $\frac{A}{\hbar} \gg |\omega_1 - \omega_2|, |\omega_1 + \omega_2|$. What are the energy levels to first nonvanishing order in ω_1 and ω_2? Show that these reproduce the results for the spin-spin interaction Hamiltonian.

Problem 2

Suppose we take a special case of the Hamiltonian from the previous problem, with $\omega = \omega_1 = -\omega_2$ and $A = 0$, so that $\hat{H} = \omega(\hat{S}_{1z} - \hat{S}_{2z})$. At time $t = 0$ the two particles are in the singlet state with 0 total spin angular momentum.

(a) Find the state as a function of time. Show that the total spin of the system oscillates between spin-0 and spin-1, and find the period of oscillation.

(b) At time t, measurements are made of S_{1y} and S_{2y}. Calculate the probability that both yield the value $\hbar/2$.

Problem 3

A system with two subsystems of fixed spin can be written in two bases, either $|j_1, m_1\rangle_1 |j_2, m_2\rangle_2$, or $|j, m\rangle$, where j ranges from $j_1 + j_2$ to $|j_1 - j_2|$.

(a) Show that these two bases have the same number of elements. This is necessary since the number of elements of any basis of a vector space should be the same.

(b) How many Clebsch-Gordon coefficients are there in this case? Explain why many of them will be 0.

Problem 4

Derive the Clebsch-Gordon coefficients for addition of a spin-$\frac{1}{2}$ particle and a spin-$\frac{3}{2}$ particle $\langle j, m|(|\frac{1}{2}, m_1\rangle_1|\frac{3}{2}, m_2\rangle_2)$, with $m > 0$. Explain how to get the remaining ones.

Note: Once you have any state $|j, m\rangle$ in the desired basis, you can apply raising/lowering operators to get more states. In addition, you can use orthonormality to find states. With both of these you can find all the Clebsch-Gordon coefficients starting from any state.