Assignment 5

171.303 Quantum Mechanics I

Due: October 16, 2018

Problem 1

(a) A spin-2 particle has 5 states, which can be chosen as $|2, m_z\rangle$ for $m_z = 2, 1, 0, -1, -2$. In this basis, find the representation of $\hat{S}_x$.

(b) Check that the vector

$$|2, m_x\rangle \to \frac{1}{4}\begin{pmatrix} 1 \\ 2 \\ \sqrt{6} \\ 2 \\ 1 \end{pmatrix}$$

is an eigenvector of $\hat{S}_x$. What is m_x for this eigenstate?

Problem 2

For any state, define an angle $0 \leq \theta \leq \pi$ via

$$\cos\theta = \frac{\langle J_z \rangle}{\langle |\mathbf{J}| \rangle}.$$

(a) Find the minimum value of θ for a spin-j particle.

(b) Give a rough interpretation of the value calculated in part (a). What limit corresponds to a classical spinning top? What does your result from part (a) say in this limit?

(c) How large must j be to allow a value of $\theta = 1°$? What is the corresponding total angular momentum, expressed in SI units?

Problem 3

We have a 3-state system with Hamiltonian represented by the matrix

$$\hat{H} \to \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{pmatrix}$$

with orthonormal basis states $|1\rangle, |2\rangle, |3\rangle$, where a, b, c are real constants with dimensions of energy.

(a) What are the possible energies of the system?

(b) Suppose our system is initially in the state $|\psi(0)\rangle = |2\rangle$. Find $|\psi(t)\rangle$. Calculate $\langle E\rangle$ explicitly as a function of time.

(c) Calculate the probabilities of measuring the state in each of the states $|1\rangle, |2\rangle, |3\rangle$ as functions of time.

Problem 4

A spin-$\frac{1}{2}$ particle enters a region with an inhomogeneous magnetic field. It experiences a time-dependent Hamiltonian given by

$$\hat{H}(t) = \omega e^{-at^2} \hat{S}_z$$

where ω, a are real constants with a positive.

(a) Calculate the time evolution operator $\hat{U}(\infty, -\infty)$ which (by definition) satisfies

$$\hat{U}(\infty, -\infty)|\psi(-\infty)\rangle = |\psi(+\infty)\rangle.$$

Express it as a rotation operator and find its representation in the S_z basis.

Note: You can assume the result from problem 4.2 in the textbook. You may find results from Appendix D useful.

(b) We perform an experiment by shooting particles in the state $|\psi(-\infty)\rangle = |+\mathbf{x}\rangle$ through the magnetic field. At a late time $t \to \infty$ we measure the angular momentum of the particles along the y axis and find that 100% of them have $S_y = -\frac{\hbar}{2}$. What are the possible values of ω in terms of a?

(c) Do you expect energy to be conserved at all times for this Hamiltonian? Check your answer by considering a state $|+\mathbf{z}\rangle$ at time $t = 0$. What does this evolve to at very late times $t \to \infty$? What is $\langle E(t)\rangle$ at times $t = 0$ and $t \to \infty$ for this state?