Assignment 4

171.303 Quantum Mechanics I

Due: October 9, 2018

Problem 1

(a) Using the basis of spin-1 states $|1,1\rangle, |1,0\rangle, |1,-1\rangle$, compute the matrix representation of the operators $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$, using general principles which hold for every value of spin. Check that your results reproduce (3.28).

(b) In this representation, compute the commutator $[\hat{J}_x, \hat{J}_y]$. Relate this to $\hat{J}_z$.

(c) Compute the matrix representation of $\hat{\mathbf{J}}^2$. Show that this commutes with each of the individual generators of angular momentum.

Problem 2

Suppose we take an arbitrary (Euclidean) unit vector in 3 dimensions:

$$\mathbf{n} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}.$$

In this problem we will find an eigenvector of spin in the direction $\mathbf{n}$ for a spin-$\frac{1}{2}$ particle.

(a) The rotation operator along the y-axis is $\hat{R}(\theta\mathbf{j}) = e^{-i\hat{S}_y\theta/\hbar}$. Show that this may be expressed as

$$\hat{R}(\theta\mathbf{j}) = \cos\frac{\theta}{2} - \frac{2i}{\hbar}\hat{S}_y\sin\frac{\theta}{2}.$$

Hint: Try to find a way to write higher powers of $\hat{S}_y$ (e.g. $\hat{S}_y^2, \hat{S}_y^3, \ldots$) in terms of 1 and $\hat{S}_y$. You may find it helpful to look at a matrix representation of the operator.

(b) Apply the operator $\hat{R}(\theta\mathbf{j})$ to the state $|+\mathbf{z}\rangle$. This should rotate the spin projection by an angle θ in the xz-plane, giving you a new eigenvector $|+\mathbf{n}'\rangle$, where $\mathbf{n}' = \sin\theta\mathbf{i} + \cos\theta\mathbf{k}$.

(c) Now, take the vector $|+\mathbf{n}'\rangle$ from part (b) and apply the rotation operator $\hat{R}(\phi\mathbf{k}) = e^{-i\hat{S}_z\phi/\hbar}$. Show that the resulting vector you get is (up to an overall phase):

$$|+\mathbf{n}\rangle = \cos\frac{\theta}{2}|+\mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\mathbf{z}\rangle.$$

Note: You don't need the result from (a) for this part; there is a much easier approach!

Problem 3

Consider a spin-$\frac{1}{2}$ particle in the state

$$|\psi\rangle = \frac{1}{2}|+\mathbf{z}\rangle - \frac{\sqrt{3}}{2}e^{-\frac{\pi}{4}i}|-\mathbf{z}\rangle.$$

(a) Compute the expectation values $\langle S_x\rangle, \langle S_y\rangle$, and $\langle S_z\rangle$.

(b) The equation $[\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$ leads to an uncertainty relation. Write down this relation, compute the quantities involved, and verify directly that it holds for this state.

Problem 4

In the basis of eigenvectors for $\hat{J}_z$ for a spin-1 particle, we can express the eigenvector of $\hat{J}_x$ with eigenvalue $+\hbar$ as

$$|j=1, m_x=1\rangle \rightarrow \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

(a) Using the matrix representation from (3.28), check that this is an eigenvector of $\hat{J}_x$ with the desired eigenvalue.

(b) Analogous to the raising and lowering operators along the z-axis, we can define $\hat{J}_{x,\pm} = \hat{J}_y \pm i\hat{J}_z$. Compute $\hat{J}_{x,+}|j=1, m_x=1\rangle$, and explain why your result agrees with your expectations.

(c) Using the raising operator $\hat{J}_{x,-}$, find the representations of the (properly normalized) states $|j=1, m_x=0\rangle$ and $|j=1, m_x=-1\rangle$.

(d) A spin-1 particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|j=1, m_z=-1\rangle + |j=1, m_z=0\rangle).$$

This particle is shot through a Stern-Gerlach apparatus oriented along the x-axis. Find the probability of each possible measured value of $\hat{S}_x$.