Assignment 3

171.303 Quantum Mechanics I

Due: September 28, 2018

Problem 1

A rotating disk is mounted at the origin with its axis along the z-axis, initially at rest. A beam of electrons is fired along the z-axis in the $+z$ direction. All the electrons are in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|+\mathbf{z}\rangle - \frac{i}{2}|-\mathbf{z}\rangle.$$

Suppose N electrons collide with the disk perfectly inelastically during a period of time Δt, so that all the angular momentum along the axis of rotation is transferred to the disk. Find the average torque exerted on the disk. Which direction does the disk rotate?

Problem 2

(a) Calculate the matrix representing the rotation operator $\hat{R}(\theta\mathbf{i})$ in the $|\pm\mathbf{x}\rangle$ basis.

(b) Find the change of basis matrix to go from the $|\pm\mathbf{x}\rangle$ basis to $|\pm\mathbf{z}\rangle$ basis, and use this to calculate the representation of the operator in part (a) in the $|\pm\mathbf{z}\rangle$ basis.

(c) Calculate $\hat{R}(\frac{\pi}{2}\mathbf{i})|+\mathbf{z}\rangle$. Does your result match what you would expect classically for a rotation?

Problem 3

One of the most important classes of operators in quantum mechanics are unitary operators, satisfying $\hat{U}^\dagger = \hat{U}^{-1}$. In this exercise you'll work out some of their properties.

(a) Show that unitary operators preserve inner products. That is, if $|\psi'\rangle = U|\psi\rangle$ and $|\phi'\rangle = U|\phi\rangle$, show that $\langle\psi|\phi\rangle = \langle\psi'|\phi'\rangle$. Using this fact, show that any eigenvalue λ of a unitary operator has $|\lambda| = 1$ (so that $\lambda = e^{i\theta}$ for some real value of θ).

(b) Show that the product $\hat{U}\hat{V}$ of two unitary operators is another unitary operator, and that the inverse $\hat{U}^{-1}$ of a unitary operator is a unitary operator.

Problem 4

The other important class of operators in quantum mechanics is Hermitian operators satisfying $\hat{A}^\dagger = \hat{A}$. This exercise focuses on the properties of Hermitian operators and their relationship to unitary operators.

(a) Show that if a, b are real numbers and $\hat{A}, \hat{B}$ are Hermitian operators, then $a\hat{A} + b\hat{B}$ is also a Hermitian operator. Also, show that assuming $\hat{A}$ is invertible, $\hat{A}^{-1}$ is also Hermitian.

(b) Using the facts that $e^{\hat{X}^\dagger} = (e^{\hat{X}})^\dagger$ for an arbitrary operator $\hat{X}$, and that $e^{\hat{Y}+\hat{Z}} = e^{\hat{Y}} e^{\hat{Z}}$ whenever $\hat{Z}\hat{Y} = \hat{Y}\hat{Z}$,[1] show that for $\hat{A}$ a Hermitian operator and θ a real number, $U = e^{i\hat{A}\theta}$ is a unitary operator.

[1] Both of these facts can be proven by using the series expansion for the exponential and working term-by-term to match the left and right sides, but you don't need to do so here.