Assignment 1

171.303 Quantum Mechanics I

Due: September 11th, 2018

Problem 1

Dimensional analysis is a powerful tool for getting approximate answers when exact computations are difficult or not needed. For this question, we'll apply dimensional analysis to try to understand the concept of spin in a classical framework.

(a) Classically, in SI units, the measurable quantities associated to an electron are its mass m_e and magnitude of electric charge e. In addition, we have two fundamental constants in classical electromagnetism, which we take to be c (the speed of light) and $4\pi\epsilon_0$ (the vacuum permittivity; the factor of 4π makes the description simpler in Gaussian units). Their measured values are:

$$\begin{aligned} e &= 1.602 \cdot 10^{-19} \text{ C} \\ m_e &= 9.109 \cdot 10^{-31} \text{ kg} \\ c &= 2.998 \cdot 10^{8} \text{ m/s} \\ 4\pi\epsilon_0 &= 1.113 \cdot 10^{-10} \text{ F/m} \end{aligned}$$

Using these four numbers, there is a single combination which has dimensions of length (up to an overall multiplicative constant). Write an expression for this *classical electron radius* r_e such that it has the correct units. Your expression should be of the form

$$r_e = m_e^{\alpha} e^{\beta} c^{\gamma} (4\pi\epsilon_0)^{\delta}$$

for some powers $\alpha, \beta, \gamma, \delta$.

NB: Townsend uses Gaussian units when describing electromagnetism rather than SI units. See Appendix A if you are unfamiliar with Gaussian units. To convert from Gaussian to SI units, one needs to make the replacements $e^2 \to \frac{e^2}{4\pi\epsilon_0}$ and $\frac{\mathbf{v}}{c} \to \mathbf{v}$.

(b) In quantum mechanics, we know that there is an additional fundamental constant $\hbar = 1.055 \cdot 10^{-34}$ Js. Following the same steps as in part (a), find an expression of the form

$$\alpha_{fs} = m_e^{\alpha'} e^{\beta'} c^{\gamma'} (4\pi\epsilon_0)^{\delta'}/\hbar$$

such that α_{fs} is a dimensionless number (i.e. the numerator has the same units as $\hbar$). α_{fs} is known as the *fine-structure constant*. Most physicists remember this number by its reciprocal. Calculate the numerical value of α_{fs}^{-1}.

Note: For a rigorous approach to dimensionless constants in dimensional analysis (more than you'll need here), look up the Buckingham π theorem.

(c) The measured magnetic dipole moment of an electron is close to $|\mu| = \frac{e\hbar}{2m_e}$. The quantum mechanical explanation is that the electron has an *intrinsic* angular momentum of magnitude $|\mathbf{S}| = \frac{\hbar}{2}$. Suppose instead we wanted a classical explanation for this angular momentum. View the electron as a rigid, solid sphere of radius r_e of uniform density, rotating about an axis through its center (recall the moment of inertia for such a sphere is $I = \frac{2}{5}m_e r_e^2$). At what speed must a point on the equator be moving in order to generate a rotational angular momentum of $|\mathbf{L}| = \frac{\hbar}{2}$? Express your answer in terms of c and α_{fs}. Is this a reasonable speed?

Problem 2

This problem is a review of the Lagrangian formulation of classical mechanics.

(a) Suppose we have a particle of mass m confined on a 2-dimensional system with a radial potential energy $V(r,\theta) = V(r)$. Write down its kinetic energy T in terms of the coordinates r, θ, and their time-derivatives (be sure to use the correct expression in polar coordinates).

(b) Take the potential energy to be $V(r) = kr^\alpha$. Write the Lagrangian $\mathcal{L} = T - V$, and write the Euler-Lagrange equations for this system. You do **not** need to solve these equations.

(c) Compute the canonical momenta associated to these coordinates $p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}}, p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$. Which of these (if either) is a conserved quantity, i.e. $\frac{dp}{dt} = 0$? Why?

(d) The *Hamiltonian* of the system (i.e. the total energy) is $\mathcal{H} = \dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} + \dot{\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L}$. Compute this explicitly. Can you write it as a function of r, θ, p_r, and p_θ, without their time derivatives?

Note: This process of going from a Lagrangian $\mathcal{L}(q_i(t), \dot{q}_i(t); t)$ to a Hamiltonian $\mathcal{H}(q_i(t), p_i(t); t)$ is called a *Legendre transformation*. In classical mechanics it is possible to describe most systems equally well via a Hamiltonian or a Lagrangian, but in quantum mechanics the Hamiltonian approach is simpler.

Problem 3

The following matrix represents a linear operator on a 3-dimensional real vector space:

$$\Omega = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Find the eigenvalues of Ω. List these in increasing order $\lambda_1, \lambda_2, \lambda_3$.

(b) For each eigenvalue λ_i, find a corresponding eigenvector $\mathbf{v}_i$ such that $\Omega \mathbf{v}_i = \lambda_i \mathbf{v}_i$. Then, find a normalized eigenvector u_i such that $|\mathbf{u}_i|^2 = 1$.

(c) Take the 3×3 matrix U with columns $U = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. Show that $U^T U = I$, where U^T is the transpose of U and I is the 3×3 identity matrix. This is equivalent to $\mathbf{u}_i^T \mathbf{u}_j = \delta_{ij}$. (For a finite-dimensional square matrix, $U^T U = I$ implies $U U^T = I$ and so $U^T = U^{-1}$.)

(d) Now, compute $\Lambda = U^T \Omega U$.

This process is known as diagonalizing the operator Ω. Λ represents the same linear operator in the basis of the eigenvalues of Ω, and so it has a particularly simple form. In quantum mechanics, we will often need to diagonalize (typically) Hermitian operators acting on the Hilbert space of some system in order to perform calculations efficiently.

Problem 4

Suppose we consider a sequence of Stern-Gerlach experiments as described in Chapter 1. Each Stern-Gerlach apparatus will be represented by a unit vector. For instance, $\boxed{\text{SG } +\hat{\mathbf{z}}}$ will represent an S-G apparatus oriented along the z axis, which blocks the trajectory of electrons in the $|-\mathbf{z}\rangle$ state but allows those to pass in the $|+\mathbf{z}\rangle$ state.

For each of the following experiments, assume we shoot a beam of electrons from the left through the sequence of apparatuses. Make a diagram showing the fraction of the beam which passes through each subsequent apparatus compared to the first, similar to Figure 1.3 in the textbook.

(i) $\boxed{\text{SG } +\hat{\mathbf{z}}} \rightarrow \boxed{\text{SG } +\hat{\mathbf{y}}} \rightarrow \boxed{\text{SG } -\hat{\mathbf{z}}}$

(ii) $\boxed{\text{SG } +\hat{\mathbf{z}}} \rightarrow \boxed{\text{SG } +\hat{\mathbf{x}}} \rightarrow \boxed{\text{SG } -\hat{\mathbf{y}}} \rightarrow \boxed{\text{SG } +\hat{\mathbf{z}}}$

(iii) $\boxed{\text{SG } +\hat{\mathbf{z}}} \rightarrow \boxed{\text{SG } +\hat{\mathbf{x}}} \rightarrow \boxed{\text{SG } +\hat{\mathbf{y}}} \rightarrow \boxed{\text{SG } +\hat{\mathbf{y}}}$

(iv) $\boxed{\text{SG } +\hat{\mathbf{z}}} \rightarrow \boxed{\text{SG } +\hat{\mathbf{y}}} \rightarrow \boxed{\text{SG } -\hat{\mathbf{x}}} \rightarrow \boxed{\text{SG } +\hat{\mathbf{x}}}$