**Introduction to Quantum Mechanics 171.303**  
**Final Exam 12/18/07 9-12**

Check the attached formula pages. Start each problem on a fresh page and give detailed reasoning. Please write your name on each page and ask your proctor for clarification if the text is unclear.

**Problem 1 (15 points)**

Suppose the wave function

\[ \psi(x) = N \exp\left(-\frac{x^2}{a}\right) \]

is known to be a solution to the time independent Schrödinger equation for a particle with mass \( m \) and energy \( E \) in a potential \( V(x) \). Here \( a \) is a real constant.

(a) Determine \( N \). (5 points)

(b) Determine the potential \( V(x) \) in terms of \( E, a, \) and \( m \) and discuss the connection of your result to a well known problem in quantum mechanics. (10 points)

**Problem 2 (20 points)**

The spin state of an electron can be described as \( |+z\rangle \) at time \( t=0 \). A magnetic field of magnitude \( B \) is now applied along a direction \( \hat{n} \) which forms an angle \( \varphi \) with the \( \hat{z} \) - axis.

(a) Write an expression for the initial state of the electron spin in the \( |\pm n\rangle \) basis. (5 points)

(b) Determine the time dependence of this spin state following application of the field. (5 points)

(c) Determine the time dependence of \( \langle S_z \rangle \) and \( \langle S_n \rangle \). (10 points)

**Problem 3 (20 points)**

Consider a system of two spin angular momenta \( \hat{S}_1 \) and \( \hat{S}_2 \). Define the total angular momentum operator \( \hat{S} = \hat{S}_1 + \hat{S}_2 \).

(a) Show that it is *not* possible to find simultaneous eigenstates for \( \hat{S}^2 \) and \( \hat{S}_{1z} \). (3 points)

(b) Show that it *is* possible to find simultaneous eigenstates for \( \hat{S}^2 \) and \( \hat{S}_z \). (5 points)
Denote eigenstates of \( \hat{S}_{1z} \) and \( \hat{S}_{2z} \) by \( |m_{1}m_{2}\rangle \) and eigenstates of \( \hat{S}^2 \) and \( \hat{S}_z \) by \( |sm\rangle \) and assume the quantum numbers for the angular momenta involved are \( s_1 = \frac{3}{2} \) and \( s_2 = 1 \).

(c) What values of \( s \) are possible? (2 points)

(c) Derive expressions for all states \( |sm\rangle = |\frac{5}{2}m\rangle \) where \( m = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \) in terms of the states \( |m_{1}m_{2}\rangle \). Use raising and/or lowering operators and check your results against the attached table of Clebsch-Gordan coefficients. (10 points)

**Problem 4 (15 points)**

A particle with mass \( m \) is subject to the following potential:

\[
V(x) = -\alpha \delta(x + a) + \beta \delta(x) - \alpha \delta(x - a)
\]

Here \( \alpha, \beta, \) and \( a \) are suitably dimensioned positive constants and \( \delta(x) \) is the Dirac delta function.

(a) Write expressions for and sketch bound state wave functions for this problem with due consideration for symmetry. (5 points)

(b) Derive an equation from which the bound state energy for the odd wave function can be derived and use a graphical technique to discuss the solution in appropriate limits. (10 points)

**Problem 5 (15 points)**

A particle with mass \( m \) moves in a harmonic potential \( V(x) = \frac{1}{2} m \omega^2 x^2 \). At \( t = 0 \) the wave functions takes the form: \( |\psi(t = 0)\rangle = A (|n\rangle + |n + 1\rangle) \), where \( |n\rangle \) is an energy eigenstates in the usual notation.

(c) Determine \( A \). (3 points)

(d) Determine \( |\psi(t)\rangle \). (5 points)

(a) Determine \( \langle x \rangle \) and \( \langle p \rangle \) versus time. (7 points)

**Problem 6 (15 points)**

Use the WKB approximation to obtain an estimate of the bound state energies for a particle with mass \( m \) in the potential \( V(x) = \alpha |x| \) where \( \alpha > 0 \) is a real constant.
Formulae

Simple Harmonic Oscillator

\[
\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right)
\]

\[
\hat{p} = -i \sqrt{\frac{m\omega\hbar}{2}} (\hat{a} - \hat{a}^+) \]

\[
[\hat{a}, \hat{a}^+] = 1
\]

\[
|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^+)^n |0\rangle
\]

Raising and lowering operators

\[
\hat{S}_z |sm\rangle = \hbar \sqrt{s(s+1)-m(m+1)} |s,m\pm1\rangle
\]

Pauli matrices

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Representation of rotation operator for spin-1/2 states

\[
R(\phi \hat{n}) = 1 \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} (\hat{\sigma} \cdot \hat{n})
\]

\[
\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)
\]

WKB bound states

Two hard boundaries

\[
\int_{x_1}^{x_2} p(x)dx = (n - \frac{1}{2})\pi\hbar
\]

One hard boundary

\[
(n - \frac{1}{2})\pi\hbar
\]

No hard boundaries

A potentially useful integral:

\[
\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}
\]

A table of Clebsch-Gordan coefficients is provided