171.303 Quantum Mechanics Midterm Exam
Solutions

Problem 1

The state is $|+n\rangle$ for $n$ as in the problem. We have calculated before the vector $|+n\rangle$ for a general direction $n$ in terms of its spherical coordinates $\theta, \phi$ as

$$|+n\rangle = \cos \frac{\theta}{2} |+z\rangle + \sin \frac{\theta}{2} e^{i\phi} |-z\rangle.$$  

This can be found either by solving the eigenvalue problem, or by applying a rotation operator. So for our state $|+n\rangle = \cos \frac{\pi}{8} |+z\rangle + \sin \frac{\pi}{8} |-z\rangle$.

As a note, some people, noticing that the state should be symmetric in the interchange of the $x$ and $z$ axes, guessed the expression $|+n\rangle = A(|+x\rangle + |+z\rangle)$. Note that the normalization constant $A$ is not $\frac{1}{\sqrt{2}}$ because $\langle +x | +z \rangle \neq 0$. In fact, this expression turns out to be correct once the correct $A$ is found. However it is not immediately obvious that this is the case, and requires further justification. To see the issue, note that we could have taken instead $A_\alpha (|+z\rangle + e^{i\alpha} |+x\rangle)$. $e^{i\alpha} |+x\rangle$ describes the same physical state as $|+x\rangle$ so both expressions satisfy our naive expectation that interchanging $x$ and $z$ should not change the state. However in the full expression $\alpha$ is a relative phase and does have a physical effect. It turns out that our conventions for $|+x\rangle$ do make this the correct expression when $\alpha = 0$, essentially because the correct expression turns out to be when $|A_\alpha|^2$ is maximized which happens to occur at $\alpha = 0$. It is ultimately more work to prove this than it is to simply derive the correct expression. To see the failure more concretely, suppose instead we took $n$ to lie in the $xy$ plane $45^\circ$ between the axes. Then the naive expression $A(|+x\rangle + |+y\rangle)$ is a completely different state from the correct one $\frac{1}{\sqrt{2}}(|+z\rangle + e^{i\pi/4} |-z\rangle)$. To get the correct state we must add a relative phase to $|+y\rangle$.

(a)  

$$\langle S_z \rangle = \frac{\hbar}{2}(\cos^2 \frac{\pi}{8} - \sin^2 \frac{\pi}{8}) = \frac{\hbar}{2} \cos \frac{\pi}{4} = \frac{\hbar}{2\sqrt{2}}.$$  

(b)  

$$\langle S_x \rangle = \left(\cos \frac{\pi}{8} \sin \frac{\pi}{8}\right) \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \left(\begin{array}{c} \cos \frac{\pi}{8} \\ \sin \frac{\pi}{8} \end{array}\right) = \hbar \sin \frac{\pi}{8} \cos \frac{\pi}{8} = \frac{\hbar}{2} \sin \frac{\pi}{4} = \frac{\hbar}{2\sqrt{2}}.$$
For a spin-$\frac{1}{2}$ particle, $\langle S^2_z \rangle$ is always $\frac{\hbar^2}{4}$ regardless of the state. So then

$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\frac{\hbar^2}{8}} = \frac{\hbar}{2\sqrt{2}}.$$ 

(c) 

$$\langle S_y \rangle = \left( \cos \frac{\pi}{8} \sin \frac{\pi}{8} \right) \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right) \left( \begin{array}{cc} \cos \frac{\pi}{8} \\ \sin \frac{\pi}{8} \end{array} \right) = 0.$$ 

Hence $\Delta S_y = \sqrt{\langle S_y^2 \rangle} = \frac{\hbar}{2}.$

(d) The uncertainty relation is $\Delta S_x \Delta S_y \geq \frac{\hbar}{2} |\langle S_z \rangle|$ which comes from $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z.$ Putting in the numbers we see that

$$\Delta S_x \Delta S_y = \frac{\hbar}{2\sqrt{2}} \frac{\hbar}{2\sqrt{2}} = \frac{\hbar^2}{4\sqrt{2}}$$

$$\frac{\hbar}{2} |\langle S_z \rangle| = \frac{\hbar}{2} \frac{\hbar}{2\sqrt{2}} = \frac{\hbar^2}{4\sqrt{2}}$$

$$\Delta S_x \Delta S_y \geq \frac{\hbar}{2} |\langle S_z \rangle|.$$ 

(e) The fraction that exit with $S_z = -\hbar/2$ is the probability for any individual one, which can be calculated as

$$P(S_z = -\hbar/2) = |\langle -z| + n \rangle|^2 = \sin^2 \frac{\pi}{8} = \frac{1 - \cos \frac{\pi}{4}}{2} = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{2}} \right).$$ 

Several equivalent expressions can be derived by standard algebraic manipulations.

**Problem 2**

(a) The raising operator satisfies $\hat{S}_+ \left| \frac{3}{2}, m \right> = \sqrt{\frac{15}{4} - m(m + 1)} \hbar \left| \frac{3}{2}, m + 1 \right>.$ Hence the matrix is

$$\hat{S}_+ \rightarrow \hbar \left( \begin{array}{cccc} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{array} \right).$$ 

The lowering operator is the adjoint of the raising operator so it is
\[ \hat{S} - \rightarrow \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} . \]

Then we have

\[ \hat{S}_x = \frac{\hat{S}_+ + \hat{S}_-}{2} \rightarrow \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} . \]

(b) The probability is \(|\langle x|\frac{3}{2}, -\frac{1}{2}\rangle\rangle |^2\). We must find the \(S_z\) basis representation of \(|\frac{3}{2}, -\frac{1}{2}\rangle\rangle_x\) to calculate this. The relevant equations are

\[ \hat{S}_x |\frac{3}{2}, -\frac{1}{2}\rangle\rangle_x \rightarrow \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \]

Solving any three of these equations (one is redundant) we find that

\[ b = -a/\sqrt{3} \]
\[ c = -a/\sqrt{3} \]
\[ d = a \]

Hence

\[ |\frac{3}{2}, -\frac{1}{2}\rangle\rangle_x \rightarrow a \begin{pmatrix} 1 \\ -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1 \end{pmatrix} \]

We must work with a normalized vector so we pick \(a = \sqrt{3}/8\). The probability is then

\[ |\langle x|\frac{3}{2}, -\frac{1}{2}\rangle\rangle_{z}|^2 = |a|^2 = 3/8. \]

Problem 3

(a) Letting \(n = \frac{3}{5}i + \frac{4}{5}k\), we see that \(\hat{H} = \frac{2}{\hbar} A\hat{S}_n\). The eigenvalues of \(\hat{S}_n\) are \(\pm \hbar/2\) so the eigenvalues of \(\hat{H}\) are \(\pm A\). We could also find these by looking at the matrix for \(\hat{H}\) and finding the roots of the characteristic polynomial.
The eigenvectors require a bit of computation. For $|+A\rangle$ we have

$$A \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix}.$$ 

From the first equation (the second is redundant) we see that $a = 3b$, so the (normalized) vector is

$$|+A\rangle = \frac{1}{\sqrt{10}} (3|z\rangle + |-z\rangle).$$

For $|-A\rangle$ we follow a similar procedure.

$$A \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = A \begin{pmatrix} c \\ d \end{pmatrix}$$

Hence $d = -3c$, so the (normalized) vector is

$$|-A\rangle = \frac{1}{\sqrt{10}} (|+z\rangle - 3|-z\rangle).$$

Note that it is actually not crucial here to normalize these vectors for part (b) if you are careful but it is still good practice to do so.

(b) The time-dependent state is $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |+z\rangle$. To evaluate this we express $|+z\rangle$ in the energy basis.

$$e^{-i\hat{H}t/\hbar} |+z\rangle = e^{-i\hat{H}t/\hbar} \frac{1}{\sqrt{10}} (3|+A\rangle + |-A\rangle)$$

$$= \frac{1}{\sqrt{10}} (3e^{-iAt/\hbar} |A\rangle + e^{iAt/\hbar} |-A\rangle).$$

At this point we are ready to calculate the probability.

$$P(t) = |\langle +z |\psi(t)\rangle|^2 = \frac{1}{10} |3e^{-iAt/\hbar} (|+z\rangle + A) + e^{iAt/\hbar} (|+z\rangle - A)|^2$$

$$= \frac{1}{10} |9e^{-iAt/\hbar} + \frac{1}{\sqrt{10}} e^{iAt/\hbar}|^2$$

$$= \frac{1}{100} |9 \cos \frac{At}{\hbar} - 8i \sin \frac{At}{\hbar}|^2$$

$$= \frac{1}{25} (25 \cos^2 \frac{At}{\hbar} + 16 \sin^2 \frac{At}{\hbar})$$

$$= 1 - \frac{9}{25} \sin^2 \frac{At}{\hbar}.$$ 

Many equivalent expressions can be found via algebraic manipulations.
(c) It is evident that this occurs when $\frac{A\hbar}{\pi} = n\pi$ for integer $n$ based on our calculation.

We could also have derived the same result even without solving part (b). Note that the time evolution operator $e^{-iHt/\hbar}$ is a rotation operator about the $n$ axis by an angle $2At/\hbar$. A rotation will rotate the spin polarization direction from $k$ to something else other than that, which means that the state is not purely $|+z\rangle$ but has some $|-z\rangle$ component. As such the probability will be less than 1 except if the rotation is by a multiple of $2\pi$, and so $At/\hbar = n\pi$ is the condition.

(d) Based on part (b) it is evident this does not occur.

We can also understand this in the context of rotations. The polarization vector will rotate about the $n$ axis and trace out a circle on the unit sphere as time passes. Since this axis is not in the $xy$ plane, the polarization will never reach the south pole of the unit sphere corresponding to the vector $-k$. That is to say, the state will never become $|-z\rangle$ but will always have some component of $|+z\rangle$, so the probability will never be 0.