Problem 1

(a) For convenience, let $\theta$ be a real number between 0 and $2\pi$ so that $a = \sin \theta$ and $b = \cos \theta$, which is possible since $a^2 + b^2 = 1$. Then the operator is

$$\hat{A} = \frac{2}{\hbar} \hat{S}_n$$

where $\mathbf{n} = \sin \theta \mathbf{j} + \cos \theta \mathbf{k}$. We know the eigenvalues and states for $\hat{S}_n$, so the eigenvalues of $\hat{A}$ are $\pm 1$. The corresponding eigenvectors can be obtained by the half-angle formulas.

$$| + \mathbf{n} \rangle = \left( \begin{array}{c} \cos \frac{\theta}{2} \\ i \sin \frac{\theta}{2} \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{1 + b \text{sign} a} \\ i \sqrt{1 - b} \end{array} \right)$$

$$| - \mathbf{n} \rangle = \left( \begin{array}{c} \sin \frac{\theta}{2} \\ -i \cos \frac{\theta}{2} \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{1 - b} \\ -i \sqrt{1 + b \text{sign} a} \end{array} \right)$$

where $\text{sign} a = |a|/a$ is the sign function.

(b) Using the above, the probability is given as

$$P = |\langle +y | + \mathbf{n} \rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1 - i) \left( \begin{array}{c} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{array} \right) \right|^2$$

$$= \frac{1}{2} (\cos \frac{\theta}{2} + \sin \frac{\theta}{2})^2$$

$$= \frac{1}{2} (1 + \sin \theta)$$

$$= \frac{1}{2} (1 + a)$$
Problem 2

(a) The probability is

\[ P = |\langle 0' | 0 \rangle|^2 \]
\[ \langle 0' | 0 \rangle = \int_{-\infty}^{\infty} \psi_{0'}^*(x) \psi_0(x) dx \]
\[ = \sqrt{\frac{m\sqrt{\omega'} \pi}{\hbar}} \int_{-\infty}^{\infty} e^{-m(\omega' + \omega_x^2)/2\hbar} dx \]
\[ = \sqrt{\frac{m\sqrt{\omega'} \pi}{\hbar}} \frac{\pi(2\hbar)}{m(\omega + \omega')} \]
\[ = \sqrt{\frac{2\sqrt{\omega'} \omega}{\omega + \omega'}} \]
\[ P = \frac{2\sqrt{\omega'} \omega}{\omega + \omega'} \]
\[ = \frac{2\sqrt{2}}{3} \]

(b) We first present the solution suggested in the hint. Writing the primed raising operator in terms of the position and momentum operators, and then back in terms of the unprimed raising and lowering operators, gives:

\[ \hat{a'}^{\dagger} = \sqrt{\frac{m\omega'}{2\hbar}} (\hat{x} - \frac{i}{m\omega} \hat{p}_x) \]
\[ = \sqrt{\frac{m\omega'}{2\hbar}} (\sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^{\dagger}) - \frac{i}{m\omega} \sqrt{\frac{m\omega}{2\hbar}} (\hat{a}^{\dagger} - \hat{a})) \]
\[ = \frac{1}{2} \left( \sqrt{\frac{\omega'}{\omega}} + \sqrt{\frac{\omega}{\omega'}} \right) \hat{a}^{\dagger} + \frac{1}{2} \left( \sqrt{\frac{\omega'}{\omega}} - \sqrt{\frac{\omega}{\omega'}} \right) \hat{a} \]
\[ \hat{a} = \frac{1}{2} \left( \sqrt{\frac{\omega'}{\omega}} + \sqrt{\frac{\omega}{\omega'}} \right) \hat{a}^{\dagger} + \frac{1}{2} \left( \sqrt{\frac{\omega'}{\omega}} - \sqrt{\frac{\omega}{\omega'}} \right) \hat{a}^{\dagger} \]

The average energy for the new Hamiltonian is then
\begin{align*}
\langle E \rangle &= \langle \hat{a}^{\dagger} \hat{a}' + 1/2 \rangle \hbar \omega' \\
&= \langle \langle 0 | \hat{a}^{\dagger} \hat{a}' | 0 \rangle + 1/2 \rangle \hbar \omega' \\
&= \left( \frac{1}{4} \left( \sqrt{\frac{\omega'}{\omega}} - \sqrt{\frac{\omega}{\omega'}} \right)^2 \langle 0 | \hat{a} \hat{a}' | 0 \rangle + \frac{1}{2} \right) \hbar \omega' \\
&= \left( \frac{1}{8} + \frac{1}{2} \right) 2 \hbar \omega \\
&= \frac{5}{4} \hbar \omega
\end{align*}

Here is an alternate solution. It is easy to check that for a harmonic oscillator in an energy eigenstate \( |n \rangle \), its average kinetic energy \( \langle \frac{p^2}{2m} \rangle \) and average potential energy \( \langle \frac{1}{2}m \omega^2 x^2 \rangle \) are equal. This is a special case of the Virial theorem in quantum mechanics, or it can be proven by standard methods with raising and lowering operators.

So in the state \( |0 \rangle \) we have \( \langle \frac{1}{2}m \omega^2 x^2 \rangle = \langle \frac{p^2}{2m} \rangle = \frac{1}{4} \hbar \omega \). Our new Hamiltonian is the same except with a factor of \( \frac{\omega'}{\omega} = 4 \) on the potential, so the average energy is \( \frac{5}{4} \hbar \omega \).

**Problem 3**

(a) We need to check that any pair of the three operators \( \hat{H}, \hat{J}^2, \hat{J}_z \) commutes. We already know that \( \hat{J}^2 \) and \( \hat{J}_z \) commute, so we just need to check the commutators with the Hamiltonian. We write the Hamiltonian as

\[
\hat{H} = a + b \hat{J}_1 \cdot \hat{J}_2 \\
= a + b \left( \hat{J}_1 + \hat{J}_2 \right)^2 - \hat{J}_1^2 - \hat{J}_2^2 \\
= a + \frac{b}{2} \hat{J}^2 - \frac{3}{4} bh^2
\]

This is a linear combination of the identity, which commutes with everything, and \( \hat{J}^2 \), which commutes with both \( \hat{J}^2 \) and \( \hat{J}_z \), so \( \hat{H} \) commutes with both of these. Hence, all three operators can be measured simultaneously. The four possible measurements we could have are
\[ J^2 = 0 \quad J_z = 0 \quad E = a - \frac{3}{4} \hbar^2 \]
\[ J^2 = 2\hbar^2 \quad J_z = \hbar \quad E = a + \frac{1}{4} \hbar^2 \]
\[ J^2 = 2\hbar^2 \quad J_z = 0 \quad E = a + \frac{1}{4} \hbar^2 \]
\[ J^2 = 2\hbar^2 \quad J_z = -\hbar \quad E = a + \frac{1}{4} \hbar^2 \]

(b) In this basis, the Hamiltonian is already diagonal. It is given by
\[
\hat{H} = \begin{pmatrix}
    a - \frac{3}{4} \hbar^2 & 0 & 0 & 0 \\
    0 & a + \frac{1}{4} \hbar^2 & 0 & 0 \\
    0 & 0 & a + \frac{3}{4} \hbar^2 & 0 \\
    0 & 0 & 0 & a + \frac{1}{4} \hbar^2 \\
\end{pmatrix}
\]
where we have ordered the basis the same as in the previous problem.

(c) The two stretched configurations are uniquely determined up to a global phase which we pick to be 1. So they are
\[
|\frac{1}{2}, \frac{1}{2}\rangle = |1, 1\rangle \\
| -\frac{1}{2}, -\frac{1}{2}\rangle = |1, -1\rangle \\
\]
Applying the lowering operator to the first equation, we derive \( ket1, 0, \) and by orthogonality we find \(|0, 0\rangle\). These are
\[
|\frac{1}{2}, -\frac{1}{2}\rangle + | -\frac{1}{2}, \frac{1}{2}\rangle = \sqrt{2}|1, 0\rangle \\
|\frac{1}{2}, -\frac{1}{2}\rangle - | -\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle + |0, 0\rangle) \\
|\frac{1}{2}, -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle - |0, 0\rangle)
\]

(d) We write
\[
2\hat{J}_1 \cdot \hat{J}_2 = 2(\hat{J}_{1x}\hat{J}_{2x} + \hat{J}_{1y}\hat{J}_{2y} + \hat{J}_{1z}\hat{J}_{2z}) \\
= 2\hat{J}_{1z}\hat{J}_{2z} + (\hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+})
\]
Hence the Hamiltonian is \( \hat{H} = a + b\hat{J}_{1z}\hat{J}_{2z} + \frac{h}{2} (\hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}) \). This is represented by
\[ \hat{H} = \begin{pmatrix} a + \frac{1}{4} \hbar^2 & 0 & 0 & 0 \\ 0 & a - \frac{1}{4} \hbar^2 & \frac{1}{2} \hbar^2 & 0 \\ 0 & \frac{1}{2} \hbar^2 & a - \frac{1}{4} \hbar^2 & 0 \\ 0 & 0 & 0 & a + \frac{1}{4} \hbar^2 \end{pmatrix} \]

Problem 4

(a)

\[ 1 = \langle \psi | \psi \rangle \]
\[ = \int_{-a/2}^{a/2} |\psi(x)|^2 \, dx \]
\[ = |N|^2 \int_{0}^{a/2} \left( \frac{a}{2} - x \right)^2 \, dx \]
\[ = |N|^2 \frac{a^3}{24} \]
\[ N = \sqrt{\frac{12}{a^3}} \]

(b) The first eigenstate is given by:

\[ \langle \psi | 1 \rangle = \int_{-a/2}^{a/2} \sqrt{\frac{2}{a}} \cos \frac{\pi x}{a} \frac{N(a/2 - |x|)}{a} \, dx \]
\[ = \frac{4\sqrt{6}}{a^2} \int_{0}^{a/2} \cos \frac{\pi x}{a} \frac{a}{2} - x \, dx \]
\[ = \frac{4\sqrt{6}}{a^2} \frac{a^3}{\pi^2} \]
\[ = \frac{4\sqrt{6}}{\pi^2} \]
\[ P_1 = \frac{96}{\pi^4} \]

The second is:

\[ \langle \psi | 1 \rangle = \int_{-a/2}^{a/2} \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a} \frac{N(a/2 - |x|)}{a} \, dx \]
\[ = 0 \]
\[ P_2 = 0 \]
(c) These expectation values are calculated:

\[ \langle x \rangle = \int_{-a/2}^{a/2} |\psi(x)|^2 x \, dx \]
\[ = |N|^2 \int_{-a/2}^{a/2} x (\frac{a}{2} - |x|)^2 \, dx \]
\[ = 0 \]

\[ \langle x^2 \rangle = \int_{-a/2}^{a/2} |\psi(x)|^2 x^2 \, dx \]
\[ = |N|^2 \int_{-a/2}^{a/2} x^2 (\frac{a}{2} - |x|)^2 \, dx \]
\[ = \frac{24}{a^3} \int_0^{a/2} x^2 (\frac{a^2}{4} - xa + x^2) \, dx \]
\[ = \frac{24}{a^3} (\frac{a^5}{96} - \frac{a^5}{64} + \frac{a^5}{160}) \]
\[ = \frac{a^2}{40} \]

\[ \Delta x = \frac{a}{2\sqrt{10}} \]

Problem 5

A bound state must have energy \(-V_0 < E < 0\) for this potential. Set \(k = \sqrt{\frac{2m(E+V_0)}{\hbar}}\) and \(q = \sqrt{-\frac{2mE}{\hbar}}\). The solution to the Schrodinger equation, ignoring the boundary conditions, is

\[ \psi(x) = \begin{cases} 
0 & x < 0 \\
A \sin(kx) + B \cos(kx) & 0 < x < a \\
Ce^{-qx} + De^{qx} & x > a 
\end{cases} \]

For a normalizable wavefunction, we need \(D = 0\), and for continuity at \(x = 0\) we need \(B = 0\), so the wavefunction is

\[ \psi(x) = \begin{cases} 
0 & x < 0 \\
A \sin(kx) & 0 < x < a \\
Ce^{-qx} & x > a 
\end{cases} \]

We additionally have boundary conditions at \(x = a\), which don’t affect the functional form but do restrict the values of \(C\) and \(k\).

(b) Our boundary conditions require the continuity of \(\psi(x)\) and \(\psi'(x)\) at \(x = a\). Hence,
\[ A \sin(ka) = Ce^{-qa} \]
\[ Ak \cos(ka) = -Cq e^{-qa} \]

Dividing these gives

\[ \tan(ka) = -k/q \]
\[ \tan(\sqrt{\frac{2ma^2(E + V_0)}{\hbar}}) = -\sqrt{-\frac{V_0 + E}{E}} \]

(c) The right hand side has a vertical asymptote at \( E = 0 \). For there to be 2 bound states, we need the left hand side to have 2 vertical asymptotes before \( E = 0 \). The second vertical asymptote of tangent is when the argument is \( \frac{3\pi}{2} \).

Setting these equal at \( E = 0 \), we have

\[ \frac{\sqrt{2ma^2V_{\text{min}}}}{\hbar} = \frac{3\pi}{2} \]
\[ 2ma^2V_{\text{min}} = \frac{9\pi^2\hbar^2}{4} \]
\[ V_{\text{min}} = \frac{9\pi^2\hbar^2}{8ma^2} \]

For any value of \( V_0 > V_{\text{min}} \) the system will admit at least 2 bound states, and for sufficiently close values it will admit exactly 2.

Problem 6

(a)

\[ \langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{d}{dx} \psi(x) dx \]

\[ = -i\hbar \int_{-\infty}^{\infty} -\frac{x}{a^2} e^{-x^2/2a^2} dx \]

\[ = 0 \]

In the final step, we have used the fact that the integrand is odd and the region of integration is centrally symmetric. Alternatively, one can use the fact that the wavefunction is real to show that the mean momentum is 0.

(b) Because the state is an energy eigenstate, it satisfies the TISE.

\[ (-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) - E) Ne^{-x^2/2a^2} = 0 \]
This simplifies to
\[
(-\frac{\hbar^2}{2m}\frac{x^2}{a^2} - \frac{1}{a^2} + V(x) - E)Ne^{-x^2/2a^2} = 0.
\]

The factor in parentheses must be 0. Choosing the minimum of \(V(x)\) to be 0, we have \(V(x) = \frac{\hbar^2}{2ma^2}x^2\), which is a harmonic oscillator potential, and checking the energy, we are in the ground state

(c) The lowering operator is, for some real constants \(A, B\), \(\hat{a} = A\hat{x} + iB\hat{p}_x\). We know that \(\hat{a}|\psi\rangle = 0\), so it must be that \(A\hat{x}|\psi\rangle = -iB\hat{p}_x|x\rangle\). Then applying the raising operator, we have for the first excited state wave function

\[
\psi_1(x) = \langle x|1\rangle \\
= \langle x|\hat{a}^\dagger|0\rangle \\
= \langle x|(A\hat{x} - iB\hat{p}_x)|0\rangle \\
= 2A\langle x|x\rangle \\
= 2Ax\psi_0(x).
\]

So the wavefunction is just (for some normalization constant \(N_1\))

\[
\psi_1(x) = N_1xe^{-x^2/2a^2}.
\]