Problem 1 (20 points)

$|\psi\rangle$ is an eigenstate of the angular momentum operators $\hat{L}^2$ and $\hat{L}_z$ with eigenvalues of $l(l+1)\hbar^2$ and $m\hbar$, respectively. Find $\langle \hat{L}_x \rangle$ and $\langle \hat{L}^2_x \rangle$ for this state.

Problem 2 (30 points)

For a particle in a harmonic oscillator potential described by a coherent state $|\alpha\rangle$, calculate the energy uncertainty $\Delta E$.

Problem 3 (20 points)

Consider a simple harmonic oscillator with mass $m$ and frequency $\omega$. The two lowest energy eigenstates have energies $\frac{1}{2}\hbar\omega$ and $\frac{3}{2}\hbar\omega$, respectively.

(a) Calculate the expectation value of kinetic energy for the state with total energy $\frac{1}{2}\hbar\omega$. (10 points)

(b) Calculate the expectation value of potential energy for the state with total energy $\frac{3}{2}\hbar\omega$. (10 points)

Problem 4 (30 points)

A system of two spin-$\frac{1}{2}$ particles with spin operators $\hat{S}_1$ and $\hat{S}_2$ can be described by an effective Hamiltonian

$$\hat{H} = A(\hat{S}_{1z} + \hat{S}_{2z}) + B\hat{S}_1 \cdot \hat{S}_2$$

where $A$ and $B$ are real constants.
(a) Find the energy levels of $\hat{H}$. (10 points)

(b) The total angular momentum is $\hat{S} = \hat{S}_1 + \hat{S}_2$. Find the matrix representation of $\hat{H}$ in the $|S, M, s_1, s_2\rangle$ basis. (10 points)

(c) Write the $|S, M, s_1, s_2\rangle$ states in terms of the $|s_1, s_2, m_1, m_2\rangle$ basis. (10 points)

Problem 5 (40 points)
A 1-dimensional potential has an infinite wall and a $\delta$-function attractive well,

$$V(x) = \begin{cases} \infty & x < -d \\ -\alpha \delta(x) & x > -d \end{cases}$$

where $\alpha$ is a positive constant.

(a) Find an equation for the bound state energy $E$. (25 points)

(b) How does the energy of this potential compare to the case where there is only the $\delta$-function and no infinite wall? (15 points)

Problem 6 (30 points)
A system consists of 3 particles. Particles 1 and 2 both have spin $\frac{1}{2}$, while particle 3 has spin 1.

(a) Show that there are two orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ which both satisfy

$$\hat{J}^2|\psi_j\rangle = 2\hbar^2|\psi_j\rangle$$
$$\hat{J}_z|\psi_j\rangle = \hbar|\psi_j\rangle.$$ (20 points)

(b) Can the two states be distinguished by a measurement of $\hat{L}^2$, where $\hat{L} = \hat{S}_1 + \hat{S}_2$? Explain why or why not. (10 points)

Problem 7 (30 points)
The $z$-component of an electron’s spin is measured to be $-\frac{\hbar}{2}$. At time $t = 0$ a uniform magnetic field $\mathbf{B} = B_0 \mathbf{x}$ is switched on.

(a) What would be the results of a measurement of the $z$-component of the electron’s spin at a time $T$ later? (15 points)

(b) What would be the results if we instead measure the $x$ component of spin? (15 points)