171.303 Quantum Mechanics Final Exam

December 13, 2014 2:00 - 5:00

Start each problem on a fresh page. Explain your reasoning clearly and completely for each problem. You may use your copy of Townsend's textbook. No other sources are allowed. You should cite which equations you are using if it is not clear from context. Ask your proctor for clarification if the problems are unclear.

Problem 1 (30 points)
A particle of mass $m$ is in the 1-dimensional potential well with

$$V(x) = \begin{cases} \frac{kx^2}{2} & x > 0 \\ \infty & x < 0 \end{cases}$$

where $k$ is a positive constant.

(a) Find the energy eigenvalues. (20 points)

(b) Find the wavefunction of the two lowest energy states. (10 points)

Problem 2 (30 points)
A particle of mass $m$ is in a 1-dimensional potential

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{elsewhere} \end{cases}$$

and has wavefunction at time $t = 0$ given by

$$\phi(x, t = 0) = A \left( 1 + \cos \frac{\pi x}{a} \right) \sin \frac{\pi x}{a}$$

where $A = \sqrt{8/3a}$.

(a) Find the wavefunction of the particle at a subsequent time $t = t_0$. (15 points)
(b) Find the average energy of the particle at time \( t = 0 \) and at time \( t = t_0 \). (15 points)

Problem 3 (35 points)

A particle of mass \( m \) is confined to a 1-dimensional potential well, which is an infinite square well with a delta function well at the center. The potential is given by

\[
V(x) = \begin{cases} 
-\frac{\alpha \hbar^2}{2m} \delta(x) & |x| < a/2 \\
\infty & |x| > a/2 
\end{cases}
\]

where \( \alpha \) is a positive constant with dimensions of inverse length.

(a) Suppose this well admits a bound state with \( E < 0 \). Derive a transcendental equation for \( E \) which must be satisfied by the bound state energy. Your equation should be something of the form

\[
\beta \sqrt{-E} = \tanh(\gamma \sqrt{-E})
\]

for some constants \( \beta \) and \( \gamma \). (25 points)

(b) What is the minimum value of \( \alpha \) for which such a bound state exists? What is the corresponding wavefunction (up to a normalization factor) for this value of \( \alpha \)? (10 points)

Problem 4 (40 points)

A system consists of three particles. Particle 1 is spin-1, while particles 2 and 3 are both spin-\( \frac{1}{2} \). The total angular momentum is \( \hat{J} = \hat{S}_1 + \hat{S}_2 + \hat{S}_3 \). The system is in a simultaneous eigenstate of \( \hat{J}^2 \) and \( \hat{J}_z \):

\[
\hat{J}^2 |\psi\rangle = 6\hbar^2 |\psi\rangle \\
\hat{J}_z |\psi\rangle = \hbar |\psi\rangle
\]

(a) Write the state \( |\psi\rangle \) as a linear combination of \( |1, m_1\rangle_1 |\frac{1}{2}, m_2\rangle_2 |\frac{1}{2}, m_3\rangle_3 \) basis states. (20 points)

(b) Let \( \hat{L} = \hat{S}_2 + \hat{S}_3 \) be the sum of angular momenta for particles 2 and 3. What are the possible simultaneous eigenstates \( |l, m_l\rangle \) of \( \hat{L}^2 \) and \( \hat{L}_z \) that we can measure in the state \( |\psi\rangle \)? What is the probability for each? (10 points)

(c) Again starting with the state \( |\psi\rangle \), we first measure particle 1, and find it to be in the state \( |1, 1\rangle \). What are the states we can measure particles 2 and 3 in now, and what are the respective probabilities? (10 points)
Problem 5 (30 points)
We have a large number of spin-$\frac{1}{2}$ particles, all in the same state $|\psi\rangle$. We take some fraction and measure the probability to be $P_+$ of finding the particle in the $|+z\rangle$ state.

(a) Determine the range of possible values for $\langle S_x \rangle$. (20 points)

(b) Let $n = \sin \theta \mathbf{i} + \cos \theta \mathbf{k}$, where $\mathbf{i}$ and $\mathbf{k}$ are the unit vectors in the $x$ and $z$ directions. For which values of $\theta$ (with $0 \leq \theta \leq 2\pi$) is it possible that $|\psi\rangle$ is a state of definite $\hat{S}_n$? (10 points)

Problem 6 (35 points)
A particle of mass $m$ is confined in a steady state of a 1-dimensional potential $V(x)$. Its wavefunction is

$$\psi(x) = A xe^{-bx^2}$$

where $A$ is a normalization constant and $b$ is some positive constant.

(a) Find the functional form of the potential energy $V(x)$ and the total energy relative to the minimum of $V(x)$. (20 points)

(b) Find the wavefunctions (up to a normalization factor) of the two states with energy closest to that of $\psi(x)$. (15 points)