1. (15 points) Note $A$ is proportional to $\sigma_x$, $B$ is $\sigma_x + I$ (the identity matrix), and $C$ is proportional to $\sigma_z$. Thus, $A$ and $B$ commute.

   (a) $A$ and $B$.
   (b) $A$: 3 and $-3$
       $B$: 2 and 0
       $C$: 2 and $-2$

2. (35 points) The trick is to write $\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$.

   (a) For small $\delta t$, the time evolution operator is $U(\delta t) \simeq 1 - i(\omega\delta t/2\hbar)(\hat{S}_+ + \hat{S}_-)$. The state at $t = 0$ is $|\psi(0)\rangle = |s, +s\rangle$. At time $\delta t$,

   \[
   |\psi(\delta t)\rangle = |s, +s\rangle - (i\omega\delta t/2\hbar) \hat{S}_- |s, +s\rangle \\
   = |s, +s\rangle - (i\omega\delta t/2) \sqrt{(s(s + 1) - s(s - 1)} |s, +(s - 1)\rangle
   \]

   The probability of being in the state $|s, +(s - 1)\rangle$ is the squared probability amplitude, or $s(\omega \delta t)^2/2$.

   (b) To get to the $|s, -s\rangle$ state, the lowering operator needs to be applied $2s$ times. Thus, the probability amplitude is proportional to $(\delta t)^{2s}$, and the probability, $(\delta t)^{4s}$
3. (50 points) As a shortcut, one can see that the Hamiltonian is of the form \((2A/\hbar)(\cos \theta \hat{S}_z + \sin \theta \hat{S}_x)\), where \(\theta\) is the angle of a 3-4-5 triangle. Thus, it is of the form \(\hat{H} = (2A/\hbar)\hat{S}_n\).

(a) \(\pm A\)

(b) Solve the eigenvalue equation:

\[
A \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm A \begin{pmatrix} a \\ b \end{pmatrix}
\]

The eigenstates for eigenvalues \(\pm A\) are

\[
\frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}
\]

respectively.

(c) The evolution operator can be written as

\[
U(t) = \cos(At/\hbar) - i [(4/5)\sigma_z + (3/5)\sigma_x] \sin(At/\hbar).
\]

Thus, the time evolution of the state \(|+z\rangle\) is

\[
\begin{pmatrix}
\cos(At/\hbar) - i(4/5)\sin(At/\hbar) & -i(3/5)\sin(At/\hbar) \\
-i(3/5)\sin(At/\hbar) & \cos(At/\hbar) + i(4/5)\sin(At/\hbar)
\end{pmatrix}
\begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

Thus, the probability of finding \(S_z\) of \(-\hbar/2\) is \(|-i(3/5)\sin(At/\hbar)|^2 = (9/25) \sin^2(At/\hbar)\).

(d) \(\pm 2A, 0\).

(e) The given rotation matrix will change an eigenstate of \(\hat{S}_z\) to an eigenstate of \((\cos \theta \hat{S}_z + \sin \theta \hat{S}_x)\). Thus, the columns of the rotation matrix are in fact the eigenstates of the Hamiltonian for \(\cos \theta = 4/5\) and \(\sin \theta = 3/5\).