1. (30 points) To find the probability \( P = |\langle x | 2, 0 | 2, +2 \rangle_x|^2 \), we should find \( |2, 0\rangle_x \) in the \( z \) basis. To do so, we need to find the matrix representation of \( \hat{S}_x \) and find the eigenvector associated with the \( m = 0 \) eigenvalue. We can build \( \hat{S}_x \) from the raising and lowering operators, since

\[
\hat{S}_x = (\hat{S}_+ + \hat{S}_-)/2
\]

(1)

We can build the raising and lowering operator matrices from the relation \( \hat{S}_+ |s, m\rangle = \sqrt{s(s+1) - m(m+1)} \hbar |s, m+1\rangle \). For example,

\[
\hat{S}_+ |2, -2\rangle = \sqrt{2(2+1) - (-2)(-2+1)} \hbar |2, -1\rangle.
\]

(2)

The full matrix can be found:

\[
\hat{S}_x \rightarrow \hbar \begin{pmatrix}
0 & 2 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 & 0 \\
0 & 0 & 0 & \sqrt{6} & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

(3)

and thus, noting that \( \hat{S}_- = \hat{S}_+^\dagger \), the eigenvalue equation becomes

\[
\hat{S}_x |2, 0\rangle_x \rightarrow \hbar \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & \sqrt{3/2} & 0 & 0 \\
0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\
0 & 0 & \sqrt{3/2} & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
a \\
b \\
c \\
d \\
e
\end{pmatrix} = 0.
\]

(4)

Solving for the eigenvector, we get:

\[
\begin{cases}
b = 0 \\
a + \sqrt{3/2}c = 0 \\
\sqrt{3/2}b + \sqrt{3/2}d = 0 \\
\sqrt{3/2}c + e = 0 \\
d = 0
\end{cases} \rightarrow \frac{1}{2} \begin{pmatrix}
\sqrt{3/2} \\
0 \\
-1 \\
0 \\
\sqrt{3/2}
\end{pmatrix},
\]

(5)

where we have normalized the vector. Thus, the probability is

\[
P = \left| \frac{1}{2} \begin{pmatrix}
\sqrt{3/2} \\
0 \\
-1 \\
0 \\
0
\end{pmatrix} \right|^2
\]

(6)

\[
= \frac{3}{8}
\]

(7)
2. (35 points)

(a) Yes, because \( \hat{S}^2 \) commutes with all components of \( \hat{S} \) and, e.g., \( [\hat{S}^2, \hat{S}_x \hat{S}_y] = \hat{S}_x [\hat{S}^2, \hat{S}_y] + [\hat{S}^2, \hat{S}_x] \hat{S}_y \).

(b) Plugging in the Pauli matrices, \( \hat{S}_x, \hat{S}_y \rightarrow (\hbar/2)\sigma_x, \sigma_y \), we see that \( \hat{H} \rightarrow 0 \), and therefore \( |\psi(t)\rangle = e^{-i\hat{0}}|z\rangle = |z\rangle \). One can also see this by rewriting \( \hat{H} \) in terms of raising and lowering operators,
\[
\hat{H} = \frac{A}{\hbar^2} (\hat{S}_+^2 - \hat{S}_-^2) \tag{8}
\]
that both states in the spin-1/2 system are annihilated by \( \hat{H} \).

(c) We can find the matrix representation of \( \hat{H} \) for \( s = 1 \) by plugging in the spin-1 matrices. We get
\[
\hat{H} \rightarrow 2A \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \tag{9}
\]
The eigenvalues and eigenvectors can be found in the usual way. We see however that the matrix looks like \( \sigma_y \) after the middle row and column are removed, suggesting a closed form for the time-evolution matrix can be found. Namely, since
\[
\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}
\]
We can rewrite
\[
e^{-i\hat{H}t/\hbar} \rightarrow \exp \left[ -i \frac{2At}{\hbar} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \right]
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cos 2At/\hbar - i \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \sin 2At/\hbar
\]
and thus,
\[
\exp \left[ -i \frac{2At}{\hbar} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 2At/\hbar \\ 0 \\ \sin 2At/\hbar \end{pmatrix}. \tag{11}
\]

3. (35 points) One can write the \( S_z = +\hbar/2 \) state in the \( x \) basis as \( |z\rangle = (1/\sqrt{2})(|+\bar{x}\rangle + |-\bar{x}\rangle) \). Since the (spin-dependent part) of the Hamiltonian is only non-zero for the \( |-\bar{x}\rangle \) part of the state, the time evolution due to the magnetic field produces the state
\[
|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|+\bar{x}\rangle + e^{-i(\omega_0T/2)\sigma_y}|-\bar{x}\rangle)
\]
\[
\rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \cos \omega_0T/2 & -\sin \omega_0T/2 \\ \sin \omega_0T/2 & \cos \omega_0T/2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]
\[
= \frac{1}{2} \begin{pmatrix} 1 + \cos \omega_0T/2 + \sin \omega_0T/2 \\ 1 - \cos \omega_0T/2 + \sin \omega_0T/2 \end{pmatrix}. \tag{12}
\]

Thus, the minimum time to produce the state \( |-z\rangle \) is when \( (1 + \cos \omega_0T/2 + \sin \omega_0T/2)/2 = 0 \) for the smallest \( T > 0 \). That is at \( \omega_0T/2 = \pi \), or \( T = 2\pi/\omega_0 \).