The problem of decomposition numbers of finite classical groups

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April 24, 2021

Modular representation theory

Let G be a finite group and ℓ a prime dividing $|G|$.

Representation theory of G in characteristic 0 is semisimple (pretty easy).

Representation theory of G in characteristic ℓ is not semisimple (very hard).

Fix $(\mathbb{K}, \mathcal{O}, \Bbbk)$ an ℓ-modular system:

- $\mathbb{K}$ a finite extension of $\mathbb{Q}_\ell$
- $\mathcal{O} \subset \mathbb{K}$ ring of integers
- $\Bbbk = \mathbb{K}/\mathcal{O}$, $\text{char } \Bbbk = \ell$

This allows us to "reduce representations mod ℓ:"

$$\begin{array}{ccccc} \underline{\text{characteristic } 0} & \rightsquigarrow & \underline{\text{integral lattice}} & \rightsquigarrow & \underline{\text{characteristic } \ell} \\ \rho \in \operatorname{Irr} \mathbb{K}G\text{-mod} & \rightsquigarrow & \Lambda_\rho \in \mathcal{O}G\text{-mod} & \rightsquigarrow & \Lambda_\rho \otimes \Bbbk \in \Bbbk G\text{-mod} \end{array}$$

Decomposition numbers and decomposition matrices

Fix a prime ℓ and an ℓ-modular system $(\mathbb{K}, \mathcal{O}, \Bbbk)$. Let $\rho \in \operatorname{Irr} \mathbb{K}G$, $\phi \in \operatorname{Irr} \Bbbk G$.

Definition

The number of times ϕ appears as a composition factor of ρ is called the *decomposition number* $d_{\rho,\phi}$.

Definition

The *decomposition matrix* of G is the matrix with rows labeled by $\{\rho \in \operatorname{Irr} \mathbb{K}G\}$, columns labeled by $\{\phi \in \operatorname{Irr} \Bbbk G\}$, with entries $d_{\rho,\phi}$.

Example $G = S_3$ and $\ell = 3$. The decomposition matrix is:

$$\begin{array}{c c} & \begin{array}{cc} (3) & (2,1) \end{array} \\ \begin{array}{c} (3) \\ (2,1) \\ (1,1,1) \end{array} & \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$$

Despite powerful methods from Lie theory, the decomposition matrix of S_n for an arbitrary prime $\ell \leq n$ is not known. Problem: how to compute ℓ-Kazhdan-Lusztig polynomials?

Decomposition matrices of finite groups of Lie type I

q a power of a prime, $\mathbf{G} \subset \mathrm{GL}_n(\overline{\mathbb{F}_q})$ connected reductive algebraic group, $F : \mathbf{G} \to \mathbf{G}$ Frobenius, $G(q) := \mathbf{G}^F$ is a *finite group of Lie type* e.g. $\mathrm{GL}_n(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$. Let ℓ be a prime s.t. $\ell \nmid q$.

Problem

Determine the decomposition matrix of $G(q)$ in characteristic ℓ.

There is a distinguished subset of $\mathrm{Irr}\,\mathbb{K}G(q)$ called *unipotent representations*, defined using geometry related to the flag variety of $\mathbf{G}$ (Deligne-Lusztig varieties).

$\#\{\mathrm{Irr}\,\mathbb{K}G(q)\} \to \infty$ as $q \to \infty$ but $\#\{\text{unip reps of } \mathbb{K}G(q)\}$ is finite, indep of q, depends only on the Weyl group W of G.

Definition

The unipotent decomposition matrix of $G(q)$ in characteristic ℓ is the submatrix of the decomposition matrix of $G(q)$ given by $D = (d_{\rho,\phi})$ s.t. ρ is unipotent, $\phi \in \mathrm{Irr}\,\Bbbk G(q)$ s.t. $d_{\rho,\phi} \neq 0$ for some unip ρ.

Expectation: the decomposition matrix of $G(q)$ can be recovered from D.

Decomposition matrices of finite groups of Lie type II

Revised problem

Determine the unipotent decomposition matrix D of $G(q)$ in char ℓ.

Example: $G(q) = \mathrm{GL}_3(q)$, $\{\text{unip reps of } \mathrm{GL}_3(q)\} \overset{1:1}{\leftrightarrow} \{\text{partitions of } 3\}$. Take $\ell \mid \Phi_3(q) = q^2 + q + 1$. Then:

$$D = \begin{array}{c c} & \begin{array}{ccc} (3) & (2,1) & (1,1,1) \end{array} \\ \begin{array}{c} (3) \\ (2,1) \\ (1,1,1) \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

Properties of the unipotent decomposition matrix D:

- the matrix D is square and invertible (Geck-Hiss),
- **expected by experts but open:** for ℓ large enough, D does not depend on q but only on the order of q mod ℓ,
- D is lower unitriangular if q is a power of a good prime for G (conj: Geck; proof: Brunat-Dudas-Taylor)
  $\rightsquigarrow$ labels for $\{\phi \in \mathrm{Irr}\, \Bbbk G(q) \mid d_{\rho,\phi} \neq 0 \text{ some unip } \rho\}$.

The order polynomial of $G(q)$ and cyclotomic polynomials

Let W be the Weyl group of $G(q)$ and assume F acts trivially on W. If $G = \mathrm{GL}_n$ then $W = S_n$, if $G = \mathrm{Sp}_{2n}$ or SO_{2n+1} then $W = B_n$. $R \subset W$ set of reflections, $m = \mathrm{rk} G$.

The order of $G(q)$ is a polynomial in q:

$$|G(q)| = q^{|R|} \prod_{i=1}^{m} (q^{d_i} - 1)$$

where d_i are the degrees of the generators of $S(\mathfrak{h})^W$.

Recall:

$$q^a - 1 = \prod_{d|a} \Phi_d(q),$$

where $\Phi_d(q)$ is the d'th cyclotomic polynomial. Thus $\ell \nmid q$ and $\ell \mid |G(q)|$ implies $\ell \mid \Phi_d(q)$ for some d dividing some d_i.

The more times $\Phi_d(q)$ divides $|G(q)|$, the more difficult to determine D in char ℓ for $\ell \mid \Phi_d(q)$.

D is known when:

- $\Phi_d(q)$ divides $|G(q)|$ exactly once (1's on diagonal, some 1's on subdiagonal, 0's else),
- $d = 1$ (identity matrix).

Methods for computing decomposition matrices of FGLT

1. Algebraic - Harish-Chandra induction/restriction to produce new characters from old, connection to Hecke algebras (1990's)
2. Geometric - Deligne-Lusztig varieties produce projective characters via cohomology, give info about cuspidals (reps that can't obtained by HC induction) (2010's)
3. Combinatorial/Lie theoretic - Kac-Moody categorification, mod d combinatorics of partitions (type A) or bipartitions (types B/C/D) (reduce, reuse, recycle)

Decomposition matrix known in type A + cases reducible to type A

- In the case of $\mathrm{GL}_n(q)$ (or $\mathrm{SL}_n(q)$, $\mathrm{PGL}_n(q)$), the decomposition matrix when $\ell \mid \Phi_d(q)$, $2 \leq d \leq n$, is given by the decomposition matrix of a q-Schur algebra for q a primitive d'th root of unity (known for $\ell >> 0$ by Ariki, Lascoux-Leclerc-Thibon...). **Parabolic affine KL polynomials of type A.**
- For $\mathrm{Sp}_{2n}(q)$ or $\mathrm{SO}_{2n+1}(q)$ and $\ell \mid \Phi_d(q)$ with d **odd**, the decomposition matrix is determined from that of $\mathrm{GL}_n(q)$ (Gruber-Hiss).

Types B and C

From now on, q is always a power of an odd prime.

$G(q) = \mathrm{Sp}_{2n}(q)$ or $\mathrm{SO}_{2n+1}(q)$. Then $W = B_n$ (hyperoctahedral group).

Order polynomial:

$$|G(q)| = q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)$$

Example: $G(q) = \mathrm{Sp}_4(q)$ or $\mathrm{SO}_5(q)$. Then

$$|G(q)| = q^4 \Phi_1(q)^2 \Phi_2(q)^2 \Phi_4(q).$$

Interesting case: $\ell \mid \Phi_2(q)$. The decomposition matrix of the principal block (the block containing the trivial representation 2.) was found by Okuyama-Waki:

$$\begin{array}{c c}
 & \begin{array}{ccccc} 2. & .2 & B_2 & 1^2. & .1^2 \end{array} \\
\begin{array}{c} 2. \\ .2 \\ B_2 \\ 1^2. \\ .1^2 \end{array} & \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot \\ 1 & 1 & 2 & 1 & 1 \end{pmatrix}
\end{array}$$

Types B and C: parametrization of unipotent representations

$G = \mathrm{SO}_{2n+1}(q)$ or $\mathrm{Sp}_{2n}(q)$. $W = B_n$.

Unipotent representations of G:

- Organized into "B_{t^2+t}-series," one series for each $t \in \mathbb{Z}_{\geq 0}$ such that $t^2 + t \leq n$.
- In the B_{t^2+t}-series, there is a unipotent representation for each bipartition $\lambda^1.\lambda^2$ of $n - (t^2 + t)$.

We write

$$B_{t^2+t} : \lambda^1.\lambda^2$$

for the unipotent representation in the B_{t^2+t}-series labeled by the bipartition $\lambda^1.\lambda^2$ of $n - (t^2 + t)$.

Example: take $n = 6$. The unipotent representations of G are parametrized by:

- Bipartitions of 6 ("principal series"), e.g. 4.1^2,
- Bipartitions of 4 (B_2-series), e.g. $B_2 : 21.1$,
- Bipartitions of 0 (B_6-series), there is only one, denoted simply as B_6.

What is the unipotent decomposition matrix in types B and C?

$G = \mathrm{SO}_{2n+1}(q)$ or $\mathrm{Sp}_{2n}(q)$, $W = B_n$.

$\ell \mid \Phi_d(q)$, d even. (The case not given by type A stuff). Assume $\ell >> 0$, so we expect some "characteristic-free" decomposition matrix only depending on d.

The unipotent decomposition matrix is lower-unitriangular with rows and columns labelled by bipartitions $B_{t^2+t} : \lambda^1.\lambda^2$.

Problem

What is the unipotent decomposition matrix in types B and C? Is there a positive combinatorial formula for the decomposition numbers in terms of the combinatorics of bipartitions? Or in terms of (parabolic, affine) Kazhdan-Lusztig polynomials?

Some issues:

- We have bipartitions of different sizes. How to relate them? Combinatorics of blocks and "cocores" suggests level-rank duality, but it's not clear how to use this.
- In type A, there is a quasihereditary algebra whose decomposition matrix gives D. But in types B and C we don't know of an algebra that could play this role.
- We don't have many examples.

In order to attack this problem we should use all available methods, both to understand the representation category better and to produce more examples.

Kac-Moody categorification in representation theory

The third method for studying decomposition numbers of classical finite groups of Lie type: Kac-Moody categorification and the combinatorics of crystals.

Chuang and Rouquier proved Broué's Abelian Defect Group Conjecture for symmetric groups in the early 2000s using **categorification**. Strategy: turn a category of representations *itself* into a representation.

Chuang-Rouquier found a way to define an action of the Lie algebra $\mathfrak{sl}_2$ on a category. **The idea:** the generators e and f of $\mathfrak{sl}_2$ act on a module category $\mathcal{C}$ by a biadjoint pair of exact endofunctors E and F in such a way that the images of the functors in the Grothendieck group of $\mathcal{C}$ satisfy the $\mathfrak{sl}_2$-relations.

Categorical actions

- yield powerful structural results such as
  - derived equivalences between blocks,
  - branching rules for induction and restriction.
- cast intricate combinatorial shadows.

Definition of $\mathfrak{g}$-categorification

$\mathfrak{g}$ a Lie algebra (finite or affine Dynkin type).
$\mathcal{C}$ – abelian category, finite-length

Definition (Chuang-Rouquier)

A $\mathfrak{g}$-categorification on $\mathcal{C}$ is a collection of exact endofunctors $\{E_i, F_i\}$ of $\mathcal{C}$, where i ranges over the nodes of the Dynkin diagram of $\mathfrak{g}$, satisfying:

- For each i, E_i and F_i are a biadjoint pair of functors;
- The functors E_i and F_i for all i induce an action of $\mathfrak{g}$ on the (complexified) Grothendieck group $[\mathcal{C}]$ via $[E_i] = e_i$, $[F_i] = f_i$ where e_i, f_i are the Chevalley generators of $\mathfrak{g}$;
- The classes $[S]$ in $[\mathcal{C}]$ of the simple objects $S \in \mathcal{C}$ are $\mathfrak{g}$-weight vectors;
- **Strong:** Set $E = \bigoplus_i E_i$, $F = \bigoplus_i F_i$. There are natural transformations $X \in \mathrm{End}(E)$ and $T \in \mathrm{End}(E^2)$ such that in $\mathrm{End}(E^n)$, $X_j := 1^{j-1}X1^{n-j}$ and $T_k := 1^{k-1}T1^{n-k-1}$ satisfy defining relations of an affine Hecke algebra.

Consequence: $\mathrm{Soc}(E_i(S))$ and $\mathrm{Head}(F_i(S))$ are simple or 0 for all simple obj S
The recipe, usually: $\mathcal{C}$ is a tower of module categories and E and F are Restriction and Induction.

Example: the symmetric groups in positive characteristic

char $\Bbbk = p > 0$, $\mathcal{C} = \bigoplus_{n \geq 0} \Bbbk\mathfrak{S}_n\text{-mod}$. $\mathrm{Res} := \bigoplus_{n \geq 0} \mathrm{Res}_n^{n+1}$, $\mathrm{Ind} := \bigoplus_{n \geq 0} \mathrm{Ind}_n^{n+1}$.

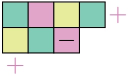
Theorem (Chuang-Rouquier, Lascoux-Leclerc-Thibon)

There is a $\widehat{\mathfrak{sl}}_p$-categorification on $\mathcal{C}$ with

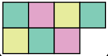
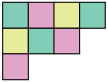
$$\mathrm{Res} = E = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} E_i, \qquad \mathrm{Ind} = F = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} F_i$$

$\lambda \vdash n$ a p-regular partition, S_λ a simple $\Bbbk\mathfrak{S}_n$-module. The head of $F_i(S_\lambda)$ is simple if $F_i(S_\lambda) \neq 0$, set it equal to $S_{\tilde{f}_i(\lambda)}$.

Example: finding $\tilde{f}_1(\lambda)$ when $\lambda = (4,3)$ and $p = 3$:



$+(-+) \rightsquigarrow +$

$\tilde{f}_1$ () = 

The $\widehat{\mathfrak{sl}}_d$-crystal

The graph with

$$\textbf{Vertices: } \{p\text{-regular partitions } \lambda\}$$
$$\textbf{Edges: } \{\lambda \to \mu \mid \mu = \tilde{f}_i(\lambda) \text{ for some } i \in \mathbb{Z}/p\mathbb{Z}\}$$

is called the $\widehat{\mathfrak{sl}}_p$-crystal on the set of p-regular partitions.

Generalizations with representation-theoretic meaning:

- can replace prime p with an integer $d \geq 2$,
- the rule for $\tilde{f}_i$ and $\tilde{e}_i$ works on any partition, not just p-regular ones,
- can replace partitions with ℓ-partitions $\lambda^1.\lambda^2.\ldots.\lambda^\ell$.

The resulting $\widehat{\mathfrak{sl}}_d$-crystals on partitions or ℓ-partitions give branching rules for induction and restriction for:

- the Hecke algebras $H_q(S_n)$ for q a d'th root of unity,
- the q-Schur algebras for q a d'th root of unity,
- groups $\mathrm{GL}_n(q)$ in characteristic ℓ when $\ell \mid \Phi_d(q)$,
- cyclotomic Hecke algebras at d'th roots of 1,
- cyclotomic rational Cherednik algebras...

$\widehat{\mathfrak{sl}}_d$-action on module categories of finite classical groups

Starting point: Gerber-Hiss-Jacon conjectured (2014) that the $\widehat{\mathfrak{sl}}_d$-crystal when d is odd describes the branching rule for Harish-Chandra induction and restriction for unipotent representations of finite unitary groups when $\ell \mid \Phi_{2d}(q)$. Proved by Dudas-Varagnolo-Vasserot (2015) by constructing a categorical KM-action.

Types B/C: $\mathrm{char}\,\Bbbk = \ell > 0$, $\ell \mid \Phi_d(q)$, $d \geq 2$ **even**.

Set $\mathcal{C}_n = \Bbbk\mathrm{Sp}_{2n}(q)\text{-mod}^{\mathrm{unip}}$ or $\Bbbk\mathrm{SO}_{2n+1}(q)\text{-mod}^{\mathrm{unip}}$, then take

$$\mathcal{C} = \bigoplus_{n \geq 0} \mathcal{C}_n$$

$\mathrm{Res}_n^{n+1} : \mathcal{C}_{n+1} \to \mathcal{C}_n$ HC restriction,
$\mathrm{Ind}_n^{n+1} : \mathcal{C}_n \to \mathcal{C}_{n+1}$ HC induction.
Then $\mathrm{Res} = \bigoplus_{n \geq 0} \mathrm{Res}_n^{n+1}$, $\mathrm{Ind} = \bigoplus_{n \geq 0} \mathrm{Ind}_n^{n+1}$ are exact, biadjoint endofunctors of $\mathcal{C}$.

Theorem (Dudas-Varagnolo-Vasserot, 2016)

There is a $\widehat{\mathfrak{sl}}_d$-categorification on $\mathcal{C}$ with

$$\mathrm{Res} = E = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} E_i \quad \textit{and} \quad \mathrm{Ind} = F = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} F_i.$$

Corollary

The Harish-Chandra branching rule for $\mathcal{C}$ is given by the $\widehat{\mathfrak{sl}}_d$-crystal on a sum of level 2 Fock spaces $\mathcal{F}_{\mathbf{s}_t}$ corresponding to the unipotent reps in the B_{t^2+t}-series:

$$[\mathcal{C}] \cong \bigoplus_{t\in\mathbb{Z}_{\geq 0}} \mathcal{F}_{\mathbf{s}_t}$$

Charge $\mathbf{s}_t \in \mathbb{Z}^2$ for level 2 Fock space $\mathcal{F}_{\mathbf{s}_t}$:

$$\mathbf{s}_t = \begin{cases} (t, -1-t+\frac{d}{2}) & \text{if } t \text{ is even,} \\ (-1-t, t+\frac{d}{2}) & \text{if } t \text{ is odd,} \end{cases}$$

Example: Say $d = 4$ and we want to determine the simple head of $F_1(S_{B_2:43.2^31})$:

$$\tilde{f}_1(B_2 : 43.2^31) = \tilde{f}_1 \left(\begin{array}{|c|c|c|c|} \hline -2 & -1 & 0 & 1 \\ \hline -3 & -2 & -1 & \\ \cline{1-3} \end{array} \qquad \begin{array}{|c|c|c} \hline 3 & 4 & + \\ \hline 2 & 3 & \\ \hline 1 & 2 & \\ \hline 0 & + & \\ \cline{1-1} \end{array} \right) = \begin{array}{|c|c|c|c|} \hline -2 & -1 & 0 & 1 \\ \hline -3 & -2 & -1 & \\ \cline{1-3} \end{array} \qquad \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & 3 \\ \hline 1 & 2 \\ \hline 0 & 1 \\ \hline \end{array}$$

The unipotent decomposition matrix of $\mathrm{SO}_{4n+1}(q)$ or $\mathrm{Sp}_{4n}(q)$ when $d = 2n$

First application of DVV's theorem to decomposition matrices:

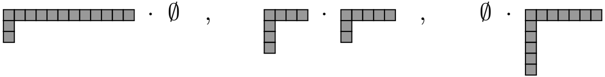
Theorem (Dudas-N., '21)

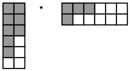
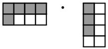
We find the unipotent decomposition matrix of $\mathrm{SO}_{4n+1}(q)$ or $\mathrm{Sp}_{4n}(q)$ when $\ell \mid \Phi_{2n}(q)$ and ℓ is sufficiently large.

The principal block is the only block of defect > 1, so we find the unipotent decomposition matrix of the principal block.

Unipotent representations in the principal block:

- certain hook bipartitions of $2n$ (principal series), such as



- bipartitions of $2n-2$ (B_2 series) fitting into certain boxes: 
- bipartitions of $2n-6$ (B_6 series) fitting into certain boxes: 

Example of our theorem when $2n = 6$

The unipotent decomposition matrix of the principal block of $\mathrm{SO}_{13}(q)$ or $\mathrm{Sp}_{12}(q)$ when $\ell \mid \Phi_6(q)$. First found by Dudas-Malle [DM '20], given by our theorem for $n = 3$.

$6.$	1																										
$.6$		1																									
$51.$	1		1																								
2.4	1	1		1																							
$B_2 : .2^2$					1																						
$41^2.$			1			1																					
$.51$		1					1																				
21.3	1		1	1				1																			
1.41		1		1			1		1																		
$B_2 : 1.21$					1					1																	
B_6											1																
$21^2.2$			1			1		1				1															
$1^2.31$				1				1	1				1														
$B_2 : 2.1^2$										1				1													
$31^3.$						1									1												
$.41^2$							1		1							1											
$B_2 : 1^2.2$										1							1										
$21^3.1$						1						1			1			1									
$1^3.21$								1				1	1						1								
$B_2 : 21.1$					1					1				1			1			1							
$B_2 : 2^2.$					1															1	1						
$1^4.1^2$												1						1	1			1					
$21^4.$															1			1					1				
$.31^3$									1				1			1								1			
$1^6.$											2							1				1	1		1		
$.21^4$									1										1					1		1	
$.1^6$																			1		2	1				1	1

Example of an argument using the crystal to find decomposition numbers

We used all three methods – algebraic, geometric, combinatorial – to prove our theorem. The combinatorics of the $\widehat{\mathfrak{sl}}_{2n}$-crystal gave us an argument showing that almost all the entries in the four "cuspidal columns" are 0 (in particular, the columns that have a 2 in them).

We also used the $\widehat{\mathfrak{sl}}_{2n}$-crystal to establish indecomposability of induced PIMs. The (unipotent part of the) characters of PIMs are the columns of the dec matrix.
Example: column $B_2 : 21.1$ in the example matrix. By Fong-Srinivasan, $P_{B_2:21.} = B_2 : 21.$. Charge $\mathbf{s}_1 = (-2, 4)$. We have:

$$F_4\left(P_{B_2:21.}\right) = F_4\left(\begin{array}{|c|c|}\hline -2 & -1 \\ \hline -3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\;.\right) = \begin{array}{|c|c|}\hline -2 & -1 \\ \hline -3 & -2 \\ \hline \end{array}\;. \;+\; \begin{array}{|c|c|}\hline -2 & -1 \\ \hline -3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\;.\;\boxed{4}$$

- this is the character of a projective module bec. F_i takes proj. to proj., but we don't know yet if it is indecomposable,
- $\tilde{f}_4(B_2 : 21.) = B_2 : 21.1 \implies P_{B_2:21.1} \mid F_4\left(P_{B_2:21.}\right)$,
- it is impossible to tell if $F_4\left(P_{B_2:21.}\right)$ is indecomposable by HC restriction!!!,
- however, we check that $\tilde{e}_i(B_2 : 2^2.) = 0$ for all $i \in \mathbb{Z}/6\mathbb{Z}$, which implies that $P_{B_2:2^2.}$ is not a summand of an induced projective module,
- so we can conclude that in fact, $F_4\left(P_{B_2:21.}\right) = P_{B_2:21.1}$.

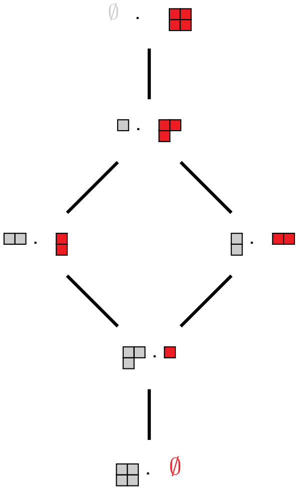
Any patterns in the decomposition matrix for the $2n = 6$ example?

Yes! Let's look at the submatrix of rows/columns labeled by the B_2 series:

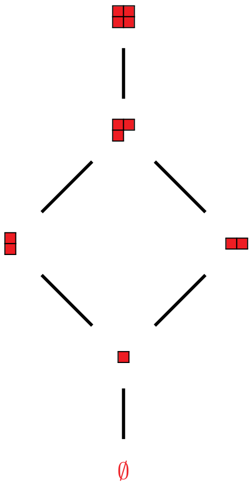
1. Consider submatrix labeled $B_2 : \lambda^1.\lambda^2$:

$$\begin{array}{r|cccccc}
B_2 : .2^2 & 1 & & & & & \\
B_2 : 1.21 & 1 & 1 & & & & \\
B_2 : 2.1^2 & & 1 & 1 & & & \\
B_2 : 1^2.2 & & 1 & & 1 & & \\
B_2 : 21.1 & 1 & 1 & 1 & 1 & 1 & \\
B_2 : 2^2. & 1 & & & & 1 & 1
\end{array}$$

2. Draw poset determined by nonzero dec numbers:



Project $\lambda^1.\lambda^2$ onto λ^2:



Question: where have we seen this poset before?

Answer:

- This is the poset of Schubert cells in the Grassmannian $\mathrm{Gr}(2, 4)$.
- This is the poset of parabolic category $\mathcal{O}^{\mathfrak{p}}$ of type $A_1 \times A_1 \subset A_3$.
- The submatrix of the decomposition matrix labeled $B_2 : \lambda^1.\lambda^2$ from 2 slides ago is the same as the decomposition matrix of $\mathcal{O}^{\mathfrak{p}}$ (multiplicities of simples in Vermas).

Some history: Category $\mathcal{O}^{\mathfrak{p}}$ of type $A_{k-1} \times A_{n-k-1} \subset A_{n-1}$ is equivalent to the category of perverse sheaves on $\mathrm{Gr}(k, n)$ (Braden, Stroppel). **Poset**: Young diagrams fitting in $k \times (n-k)$ box, under inclusion of diagrams. In our example, $k = 2$ and $n = 4$.

Why does this show up in the unip dec matrix of finite groups of Lie type in types B and C?

Brundan-Stroppel: the highest weight cover of the Hecke algebra of type B_n in the "$d = \infty$" case (parameter q generic) is equivalent to a sum of categories $\mathcal{O}^{\mathfrak{p}}$ of type $A_{k-1} \times A_{m-k-1} \subset A_{m-1}$. Explicit construction of a block as the module category of a finite-dimensional algebra called the *Khovanov arc algebra*, related to the *Temperley-Lieb algebra*. Brundan-Stroppel gave an explicit combinatorial formula for the decomposition numbers for that algebra (and thus for the Hecke algebra of type B_n when q is generic).

When $d > |\lambda^1| + |\lambda^2|$ we can expect similar behavior to $d = \infty$.

Generic submatrices of the decomposition matrix

By the B_{t^2+t}*-submatrix* we mean the submatrix of the unipotent decomposition matrix of $\Bbbk\mathrm{SO}_{2n+1}(q)$ or $\Bbbk\mathrm{Sp}_{2n}(q)$ whose rows and columns are labeled by $B_{t^2+t} : \lambda^1.\lambda^2$, where $\lambda^1.\lambda^2$ is a bipartition of $n-(t^2+t)$.

Theorem (Dudas-N. '21, work in progress)

Let $d > n - t^2 - t$ be even and let $\ell = \mathrm{char}\,\Bbbk$ be any prime such that $\ell \mid \Phi_d(q)$. Then the decomposition numbers in the B_{t^2+t}-submatrix are given by Brundan-Stroppel's formula.

Idea of proof: show that when $d > n - t^2 - t$, the dec numbers are controlled by combinatorics of $\widehat{\mathfrak{sl}}_d$-crystal.

Consequences:

- Explicit, closed, diagrammatic formulas for the entries of the B_{t^2+t}-submatrix.
- The B_{t^2+t}-submatrix depends only on the order of q mod ℓ, not on ℓ.
- The B_{t^2+t}-submatrix is the same as the decomposition matrix of Category $\mathcal{O}$ of the rational Cherednik algebra $H_{d,\mathbf{s}_t}(n - t^2 - t)$.
- All dec numbers in the B_{t^2+t}-submatrix are 0 or 1.
- This is the first result identifying large submatrices of the unip dec matrix in blocks of arbitrary complexity since the 1990s.

THANK YOU