The Shafarevich conjecture for hypersurfaces in abelian varieties
(joint work with Brian Lawrence)

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Abelian varieties in number theory

An *abelian variety* is a smooth projective algebraic variety (i.e. the solution set in complex projective space of a finite set of polynomial equations) with a group structure (where the multiplication law is defined by polynomial equations).

Topologically, an abelian variety of (complex) dimension n is a (real) $2n$-dimensional torus.

An abelian variety A over $\mathbb{Q}$ is one where the defining equations are polynomials with coefficients in $\mathbb{Q}$. In such a case the points with rational coordinates $A(\mathbb{Q})$ form a group.

Abelian varieties are very important in *arithmetic finiteness theorems*.

Mordell-Weil: The rational points $A(\mathbb{Q})$ form a finitely generated group.

Faltings 2: The rational points $A(\mathbb{Q})$ that lie in a subvariety $X \subset A$ (also satisfy some additional equations) lie in a finite union of smaller-dimensional abelian varieties contained in X.

Faltings' Theorem (Shafarevich's Conjecture)

Let n be a natural number. Let S be a finite set of primes.

We say A an abelian variety over $\mathbb{Q}$ has good reduction at p if the defining equations still define a smooth space when we reduce them modulo p.

Theorem (Faltings)

There exist finitely many abelian varieties A of dimension n over $\mathbb{Q}$, with good reduction at all primes not in S, up to isomorphism.

Same expected for many other types of varieties. Known for a few: K3 surfaces (André, She), del Pezzo surfaces (Scholl), flag varieties (Javanpeykar and Loughran), complete intersections of Hodge level at most 1 (Javanpeykar and Loughran), surfaces fibered smoothly over a curve (Javanpeykar), and Fano threefolds (Javenpeykar and Loughran).

Main Theorem

Fix a single abelian variety A over $\mathbb{Q}$ of dimension n with good reduction away from S.

We work with smooth hypersurfaces $H \subset A$ (i.e. smooth subsets defined by one additional equation with rational coefficients). We say a smooth hypersurface $H \subset A$ has good reduction at p if the solution set mod p remains smooth.

Every smooth hypersurface H represents a class $[H]$ in the Neron-Severi group of A (essentially, the degree of the equation). Fix ϕ an ample class.

Theorem 1 (Lawrence-S)

Assume $n \geq 4$. There are only finitely many smooth hypersurfaces $H \subseteq A$ representing ϕ, with good reduction at all primes not in S, up to translation.

Main Theorem ($n = 3$ case)

Consider the sequence of numbers

$$1, 5, 20, 76, 285, 1065, \ldots$$

and associated sequence of binomial coefficients

$$\binom{1+5}{1}, \binom{5+20}{5}, \binom{20+76}{20}, \binom{76+285}{76}, \binom{285+1065}{285}, \ldots$$

Theorem 2 (Lawrence-S)

Assume $n = 3$. Assume that the intersection number $\phi \cdot \phi \cdot \phi$ is not a multiple of any of the binomial coefficients on this list but the first. Then there are only finitely many smooth hypersurfaces $H \subseteq A$ representing ϕ, with good reduction at all primes not in S, up to translation.

Prior work of Lawrence and Venkatesh

Theorem (Lawrence-Venkatesh)

Fix n and d. Assume they are both sufficiently large according to some complicated formula. Fix a finite set S of primes.

Then the set of hypersurfaces $H \subset \mathbb{P}^n_{\mathbb{Q}}$, of degree d, with good reduction at all primes not in S, is not Zariski dense in the moduli space of all degree d hypersurfaces.

Same strategy + new ideas = stronger result (in a slightly different setting).

Falting's method

1. Define an arithmetic invariant (the Tate module) associated to an abelian variety A.
   - For p a prime and p^r a prime power, we define $A[p^r]$ to consist of the points of A of order dividing p^r (in the group law). As a group, $A[p^r]$ is isomorphic to $(\mathbb{Z}/p^r)^{2n}$. These points have coordinates in the algebraic closure of the rationals $\overline{\mathbb{Q}}$, so the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $A[p^r]$. We define the *Tate module* as the inverse limit $\varprojlim A[p^r]$, a Galois representation on $\mathbb{Z}_p^{2n}$.
2. Show that, for abelian varieties with good reduction outside S, the Tate module can only take finitely many values (up to isomorphism).
   - This uses an argument of Faltings and Serre which we will discuss later.
3. Show that only finitely many abelian varieties can have a given Tate module (up to isomorphism).
   - This requires showing that, for any abelian varieties A, B with isomorphic Tate modules, there is a finite-to-one algebraic map $A \to B$ (an isogeny). One also must show that there are finitely many abelian varieties defined over $\mathbb{Q}$ and isogenous to a given abelian variety, up to isomorphism.

Lawrence-Venkatesh's method

1. Define an arithmetic invariant (the Galois representation) associated to a hypersurface $H \in \mathbb{P}^n$
   - This is the Galois representation on the étale cohomology group $H^{n-1}(H, \mathbb{Q}_p)$. (We should also semisimplify the Galois representation, but ignore this.)
2. Show, for hypersurfaces with good reduction outside S, the Galois representation can only take finitely many values (up to isomorphism).
   - This uses essentially the same argument as in the abelian variety case.
3. Show that the hypersurfaces that have a given Galois representation are not Zariski dense in the moduli space of hypersurfaces.
   - This uses a completely different strategy. Unlike Faltings' argument, which uses global arithmetic information over $\mathbb{Q}$ (heights), here we only use local arithmetic information over $\mathbb{Q}_p$ (p-adic Hodge theory). So in fact we show that the hypersurfaces defined by p-adic polynomials with a given p-adic Galois representation are not Zariski dense in the moduli space of hypersurfaces. We need to show the p-adic Galois representation varies a lot between p-adic points.

How to show the Galois representation varies

Let's consider a family of varieties, like the family of all hypersurfaces in $\mathbb{P}^n$. We express this as a smooth proper map $f\colon Y \to X$ where X parameterizes the members Y_x of the family.

For any $x \in X$, the cohomology $H^i(Y_x, \mathbb{Q}_p)$ carries an action of the fundamental group $\pi_1(X)$ of X. (Even $H^i(Y_x, \mathbb{Q})$ does.)

This action controls how the Galois representation varies. Need this to be "big".

Let N be the dimension of $H^i(Y_x, \mathbb{Q})$. We obtain a representation $\rho : \pi_1(X) \to GL_N(\mathbb{Q})$. Monodromy group $= \overline{\mathrm{Im}(\rho)}$.

The Lawrence-Venkatesh method as a black box

Lawrence and Venkatesh method proves $X(\mathbb{Z}[1/S])$ is not Zariski dense given a smooth proper $f : Y \to X$, under inequalities on the Hodge numbers of $H^i(Y_x, \mathbb{Q})$, plus the assumption that ρ has *big monodromy*. (e.g. SL_N, Sp_N, SO_N sufficiently big.)

Example: $X =$ moduli space of smooth hypersurfaces in $\mathbb{P}^n_{\mathbb{Z}}$, Y=universal family, $i = n - 1$.

Want to do Noetherian induction, but lose big monodromy.

What's better about hypersurfaces in abelian varieties

Let $Y \to X$ be a family of hypersurfaces in a fixed abelian variety A, parameterized by X. (e.g. X moduli space of hypersurfaces in A.)

We can construct *many* representations of $\pi_1(X)$ from Y. Two ways:

- Fix χ a 1-dimensional character of $\pi_1(A)$. Then we have twisted cohomology $H^{n-1}(Y_x, \chi)$. The fundamental group $\pi_1(X)$ also acts on this. Take ρ_χ this representation.
- Fix m a natural number. Consider $[m] : A \to A$ the multiplication-by-m map. Then $\pi_1(X)$ acts on $H^{n-1}([m]^{-1} Y_x, \mathbb{Q})$. So does $A[m]$. These actions commute. Take ρ_χ the π_1-rep on the eigenspace of $A[m]$ with eigenvalue χ.

Suffices to show, for each nonconstant family $Y \to X$, for *some* χ, the rep ρ_χ has big monodromy. Then induct.

We do this!

How to use big monodromy

Let $N = \dim H^{n-1}(Y_x, \chi) = \phi^n$. Let $\mathbb{Q}_\chi$ =coefficient field of χ. Let $\mathbb{Q}_{\chi,p} = p$-adic completion.

For each $x \in X(\mathbb{Z}[1/S])$, $H^{n-1}(Y_x, \chi)$ is a Galois representation

- This is a representation $\rho_{\chi,x}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $GL_N(\mathbb{Q}_{\chi,p})$.
- $\rho_{\chi,x}$ is unramified away from $S \cup \{p\}$.
- For every prime ℓ not in $S \cup \{p\}$, Frobenius at ℓ acts on $\rho_{\chi,x}$.
- The characteristic polynomial of Frobenius has coefficients integers in $\mathbb{Q}_\chi$ and roots of size $\ell^{(n-1)/2}$.

Faltings-Serre: There are finitely many such (semisimple) representations.

Period Maps

Suppose instead we knew there were finitely many Hodge structures among the integral points.

We would want to show a Torelli-type theorem. (There are only finitely many $x \in X$ such that $H^{n-1}(Y_x, \chi)$ has a given Hodge structure.)

Tool: Period map from the universal cover $\tilde{X}$ of X to period domain D.

Ax-Schanuel (Bakker-Tsimerman) implies a very general Torelli-like statement. For any subvariety of D, of codimension $> \dim X$, projection to X of its inverse image in $\tilde{X}$ is not Zariski dense.

Needs monodromy to act transitively on D - big monodromy condition.

p-adic Hodge theory

Crystalline cohomology: $H^{n-1}(Y_x, \chi) \otimes \mathbb{Q}_p$ is a $\mathbb{Q}_{\chi,p}$ vector space with a (semilinear) Frobenius action. Depends only on $x \mod p$.

p-adic de Rham cohomology: For each $x \in X$ with given reduction mod p, we get a Hodge filtration on this vector space. This comes from a p-adic analytic period map $X \to D$.

p-adic Hodge theory: This vector space + filtration + Frobenius is determined up to isomorphism by the Galois representation $\rho_{\chi,x}$.

Bakker-Tsimerman theorem implies p-adic version by clever LV argument with formal power series.

Problem 1: Frobenius centralizer (easy). Problem 2: semisimplification (hard).

Geometric theorem

Fix A an abelian variety of dimension n, $Y \to X$ a smooth family of hypersurfaces in A with (ample) class ϕ. Assume:

- Y is not preserved under translation by a nontrivial element of A.
- Y is not the constant family, translated by a section of A over X.
- $n \geq 4$, or $n = 3$ and ϕ^3 not one of our binomial coefficients from earlier, or $n = 2$ and ϕ^2 neither a power of 2 greater than 4, $\binom{2k}{k}$ for $k > 2$, nor 56.

Then for some character χ, ρ_χ has monodromy containing SO_N, SL_N, or Sp_N. (In fact, for almost all χ).

Let X = Zariski closure inside moduli space of hypersurfaces in A of the set of hypersurfaces with good reduction outside S (or an irreducible component). Let Y be the quotient of the universal family of hypersurfaces on X, by any points which preserve it up to translation.

- By construction, Y is not preserved up to translation by any nontrivial point of A.
- If our desired statement is false and there are infinitely many hypersurfaces with good reduction up to translation, then Y is not the constant family up to translation.
- If the original ϕ^n was not a multiple of any of the forbidden values, then ϕ^n of the quotient is not equal to any of the forbidden values.

Because it satisfies all three assumptions, some ρ_χ has big monodromy. Then by Lawrence-Venkatesh, integral points are not Zariski dense in X. This contradicts the fact that X was the Zariski closure of the integral points, so our desired statement must be true.

Tannakian monodromy groups

Krämer and Weissauer defined a *Tannakian monodromy group* associated to a smooth variety $Z \subseteq A$.

- If you know what a perverse sheaf is: They also associate a group to a perverse sheaf on A. We view varieties as perverse sheaves by taking the constant sheaf on that variety.

Concretely, they

- proved for almost all χ, $H^i(Z, \chi) = 0$ unless $i = \dim Z$
- proved for almost all χ, $\dim H^{\dim Z}(Z, \chi) = N =$ Euler characteristic of Z times $(-1)^{\dim Z}$.
- defined an algebraic group $G_Z \subseteq GL_N$ that acts naturally on $H^{\dim Z}(Z, \chi)$.

How should we think about the Tannakian monodromy group?

Should think of G_Z as like a monodromy group, but over the "space" of possible characters χ.

The usual monodromy group of ρ_χ controls variation in the Hodge structure on $H^{n-1}(Y_x, \chi)$ as we vary a point x (and fix χ).

- Mumford-Tate group normalizes the monodromy group and the Hodge torus is constant modulo the monodromy group. If monodromy is finite, the Hodge structure is constant. If monodromy is large, Hodge structures may be totally different.
- Same idea works for Galois representations.

The Tannakian monodromy group of Y_x controls variation of the cohomology of $H^{n-1}(Y_x, \chi)$ as we vary a character χ (and fix x).

- Same theorems hold.

How do Krämer & Weissauer define the Tannakian group?

They define a suitable category of perverse sheaves.

They define an operation, *sheaf convolution*, on this category. This uses three maps $pr_1, pr_2, m : A \times A \to A$. The formula is:

$$K * L = Rm_*(pr_1^* K \otimes pr_2^* L).$$

This operation behaves like tensor product - makes the category into a symmetric monoidal abelian category.

Tannakian category: any symmetric monoidal abelian category satisfying some axioms. Theorem: These are always the category of representations of a unique pro-algebraic group.

$G_Z =$ Tannakian group associated to subcategory generated by the constant sheaf on Z, acting on representation associated to the constant sheaf on Z.

Geometric strategy

This strategy has two steps:

1. For a smooth hypersurface $H \subset A$, we prove the Tannakian monodromy group G_H contains SL_N, SO_N, or Sp_N.
   - Assuming H is not invariant under translation by any nontrivial element of A.
   - Assuming $(n, [H]^n)$ avoids list of forbidden values.
2. For any family $Y \to X$ of varieties in A, where the Tannakian monodromy group G_{Y_η} of the *generic* fiber Y_η contains SL_N, SO_N, or Sp_N, we prove that for almost all χ, the usual monodromy group of ρ_χ contains SL_N, SO_N, or Sp_N.
   - Assuming Y is not a constant family, up to translation by a section of A over X.

Proof of (2)

Key topological idea : The fundamental group of the generic fiber of any morphism is a normal subgroup of the fundamental group of the total space.

Alternate version: The monodromy group of a representation, restricted to the generic fiber of a morphism, is a normal subgroup of the monodromy group of that representation.

Strategy: Make a big Tannakian group in which G_{Y_η} and ρ_χ for generic χ are both normal subgroups.

Group theory: Normal subgroups of GL_N, GO_N, GSp_N contain SL_N, SO_N, Sp_N or are contained in scalars.

Proof of (1)

Krämer proved a series of wonderful theorems controlling the Tannakian group using the “characteristic cycle”. Especially effective for smooth varieties, hypersurfaces.

For any smooth hypersurface H, not translation-invariant, conclude: G_H contains as a normal subgroup a *simple* algebraic group acting by an irreducible *minuscule* representation.

Minuscule representations: Characters of the maximal torus appearing form a single orbit under the Weyl group.

- Classified!
- One for each simple algebraic group and character of the center of that simple group
- Example: Rep $\wedge^k$ of SL_m for $1 \leq k \leq m$.

Goal: Eliminate all pairs of a simple group w/ a minuscule representation, except classical group w/ standard representation.

Proof of (1)

Goal: Eliminate all pairs of a simple group w/ a minuscule representation, except classical group w/ standard representation.

For us, dimension of representation $= \phi^n$ and is divisible by n!

This already eliminates some cases when $n > 2$ (spin representations, exceptional groups).

Only remaining representations are $\wedge^k$ of SL_m (for $1 < k < m - 1$).

Eliminating the representations $\wedge^k$

We show (essentially) that if $G_H = SL_m$ acting by $\wedge^k$ then the Hodge structure on $H^{n-1}(H, \chi)$ is $\wedge^k$ of an m-dimensional Hodge structure.

This forces the Hodge numbers $H^{n-1}(H, \chi)$ to be certain sums of binomial coefficients in the Hodge numbers of this m-dimensional Hodge structure.

Calculate the Hodge numbers $H^{n-1}(H, \chi)$ - they are Eulerian numbers times the degree of the hypersurface.

This gives a complicated but purely combinatorial system of equations. Find all solutions by 24 pages of combinatorics + computer checks.