Geometric Langlands for ℓ-adic sheaves

Everything today is joint with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky.

Background: $X/\mathbb{F}_q$ is a smooth, proj. geom.-connected alg. curve. G split reductive, $\check{G}$ = dual group.

Langlands conj:

$$\left\{ \begin{array}{l} \text{Hecke eigenfunctions} \\ \text{on } G(\mathbb{F}_q(X)) \backslash G(\mathbb{A}) / G(\mathcal{O}) \\ \qquad\qquad\qquad \| \\ \qquad\qquad\qquad \mathrm{Bun}_G(\mathbb{F}_q) \end{array} \right\} \xleftrightarrow{\simeq} \left\{ \begin{array}{l} \text{repns of} \\ \mathrm{Weil}(X) \to \check{G}(\overline{\mathbb{Q}}_\ell) \end{array} \right\}.$$

Bun_G = moduli stack of G-bundles on X.

Various geometric versions.

One, due to Beilinson–Drinfeld

(in this form, Arinkin–Gaitsgory):

Conj: X/k char $k=0$

$\Rightarrow$ $D\text{-mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{Nilp}(LS_{\check{G}})$

compatibly with Hecke functors.

Here: $LS_{\check{G}}$ is the (derived) moduli stack

of $\check{G}$-bundles on X with ∇.

c.f.:

Arithmetic Langlands vs Geom. Langlands

$\wr\wr$ $\qquad\qquad$ $\wr\wr$

$\{S^1 \to S^1\} \simeq \mathbb{Z}$ $\qquad\qquad$ $L^2(S^1) \simeq L^2(\mathbb{Z})$.

Problem: char $k = 0$ $\overset{?}{\Rightarrow}$ no relevance to actual automorphic forms.

First issue: $LS_{\check{G}}$ is very specialized to dR setting.

Example: $\check{G} = \mathbb{G}_m$:

$$0 \to \Gamma(X, \Omega^1) \longrightarrow LS_{\mathbb{G}_m} \longrightarrow \mathrm{Jac}(X) \times \mathbb{B}\mathbb{G}_m \to 0$$

Main idea: introduce a moduli space $LS^{restr}_{\check{G}}$ in the ℓ-adic setting suitable for our purposes.

<u>Notation</u>: $k = \bar{k}$, X/k sm proj. connected

$\check{G}/\bar{\mathbb{Q}}_\ell$, G/k, $\ell \in k^\times$.

$\mathrm{Lisse}(X) :=$ (DG) category of ind-lisse ℓ-adic sheaves on X.

<u>Def</u>: S affine test scheme, a map $S \longrightarrow LS^{restr}_{\check{G}}$ is the data of a symmetric monoidal functor

$\mathrm{Rep}\,\check{G} \longrightarrow \mathrm{Lisse}(X) \otimes \mathrm{QCoh}(S)$.

~~...~~ that is right t-exact.

Concretely: ~~...~~ $S = \mathrm{Spec}\,A$, RHS = A-modules in $\mathrm{Lisse}(X)$.

Rem: If you replace $\mathrm{Lisse}(X)$ w/ $D\text{-mod}(X)$, get $LS_{\check{G}}$ from before.

Examples: 1) $\check{G} = \mathbb{G}_m$:

$$LS^{restr}_{\mathbb{G}_m} = \coprod_{\substack{\sigma \text{ rk } 1 \\ \text{lisse on } X}} \mathbb{B}\mathbb{G}_m \times H^1_{\acute{e}t}(X, \overline{\mathbb{Q}}_\ell)^{\wedge}_0 \times \Omega\mathbb{A}^1.$$

derived line.

2) $LS^{restr}_{\mathbb{G}_a} = \mathbb{B}\mathbb{G}_a \times H^1_{\acute{e}t}(X) \times \Omega\mathbb{A}^1$.

not formal.

3) $\check{G}$ gen'l : Thm

$$LS^{restr}_{\check{G}} = \coprod_{(\check{M},\sigma)/\sim} LS^{restr}_{\check{G},\sigma}$$

where $\check{M}$ is a Levi of $\check{G}$
and σ is a irreducible
$\check{M}$- local system on X

& $LS^{restr}_{\check{G},\sigma}$ "looks like" the formal completion of $LS_{\check{G}}$ along the loc.-closed substack of local systems w/ semi-simplification σ.

Conj: $Shv_{Nilp}(Bun_G) \simeq IndCoh_{Nilp}(LS^{restr}_{\check{G}})$

compatibly w/ Hecke functors.

Here: $Shv(Bun_G)$ is the category of (ind-constructible)

$\overline{\mathbb{Q}}_\ell$-sheaves on Bun_G, and Shv_{Nilp}

is the subcategory of sheaves w/

sing. support in $Nilp \subseteq T^*Bun_G$.

theory developed by Deligne, Beilinson, Saito.

in the ℓ-adic setting.

Idea to consider $Shv_{Nilp}(Bun_G)$ is

taken from the "Betti GLC" of

Ben-Zvi & Nadler.

<u>Thm</u>: $QCoh(LS_{\check{G}}^{restr}) \curvearrowright Shv_{Nilp}(Bun_G)$

(compatibly w/ Hecke functors.)

Our proof is softer than Gaitsgory's analogous argument for D-modules.

1) $\mathcal{C} = Shv_{Nilp}(Bun_G)$.

Given $x \in X$ and $V \in Rep\,\check{G}$

$\rightsquigarrow H_{V,x} : Shv_{Nilp}(Bun_G) \longrightarrow Shv_{Nilp}(Bun_G)$

upgrade (due to Nadler-Yun.)

$Rep\check{G} \otimes \mathcal{C} \longrightarrow \mathcal{C} \otimes Lisse(X)$.

variant: I finite set:

$Rep\,\check{G}^I \otimes \mathcal{C} \longrightarrow \mathcal{C} \otimes Lisse(X^I)$.

2) <u>Thm</u>: $\forall \mathcal{C}$ (mild tech condition)

this structure is $\Leftrightarrow$ $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \curvearrowright \mathcal{C}$.

Reflects the Tannakian perspective on

$\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$.

<u>Consequence</u>: Self-duality of

$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ ("regularized Verdier

duality for Bun_G").

Take $k = \overline{\mathbb{F}_q}$, $X_0 / \mathbb{F}_q$ descent of X/k.

Take $F: X \longrightarrow X$ relative Frobenius.

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr] & & X \arrow[dl] \\ & k & \end{tikzcd}$$

$F: LS^{restr}_{\check{G}} \longrightarrow LS^{restr}_{\check{G}}$ corresponding

map/$\overline{\mathbb{Q}}_\ell$

map ("pullback along $F: X \to X$").

<u>Def</u>: $LS^{arthm}_{\check{G}} := \left(LS^{restr}_{\check{G}}\right)^{F=id}$.

Think of $LS^{arthm}_{\check{G}}$ as a moduli space of Weil group rep'ns.

<u>Thm</u>: $LS^{arthm}_{\check{G}}$ is a quasi-algebraic stack (not ind-, not infinite disjoint union).

<u>Rem</u>: Scholze and Zhu also introduced a version of this stack.

Example: $X_0 = \mathbb{P}^1 \Rightarrow LS_{\check{G}}^{\text{arthm}} = \check{\mathfrak{g}}/\check{G}$.

Recall Hochschild construction: A alg

$\mathcal{M} \in A\text{-mod}$ dualizable A-mod

$\rightsquigarrow [\mathcal{M}] \in HH_\bullet(A) \simeq$

Categorically: $[\mathcal{M}] = \mathrm{tr}(\mathrm{id}_{\mathcal{M}} \circlearrowright \mathcal{M}) \in \mathrm{tr}(\mathrm{id} \circlearrowright A\text{-mod})$.

In our setting:

$$\underbrace{\mathrm{QCoh}(LS_{\check{G}}^{\text{restr}})}_{A} \circlearrowleft^{F^*} \quad \circlearrowright \quad \underbrace{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)}_{\mathcal{M}} \circlearrowleft^{F_*}$$

$$\mathrm{tr}(F^* \curvearrowright \mathrm{QCoh}(LS^{\mathrm{restr}}_{\check{G}})) \ni \mathrm{tr}(F \curvearrowright \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\mathrm{QCoh}(LS^{\mathrm{arthm}}_{\check{G}}) \ni \mathrm{Drinf}.$$

Thm-in-progress: $\Gamma(LS^{\mathrm{arthm}}_{\check{G}}, \mathrm{Drinf}) = \mathrm{tr}(F \curvearrowright \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$

$\simeq \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q))$.

Rem: Uses Cong Xue's (following V. Lafforgue) work substantially.

Slogan:

automorphic forms localize onto $LS^{\mathrm{arthm}}_{\check{G}}$ (Drinfeld's picture).

Rem: Restricted GLC $\Rightarrow$

$$\mathrm{Drinf} = \omega_{LS^{\mathrm{arthm}}_{\check{G}}}.$$

Baby Example: $X_0 = \mathbb{P}^1$. $LS^{\mathrm{arthur}}_{\check{G}} = \check{G}/\check{G}$

$$\Gamma(LS^{\mathrm{arthur}}_{\check{G}}, \mathcal{O}) = \overline{\mathbb{Q}}_\ell[\check{\Lambda}]^W.$$

vs. $\mathrm{Bun}_G(\mathbb{F}_q) = \check{\Lambda}^+$ ignoring stackyness

$$\mathrm{Fun}_c(\check{\Lambda}^+) = \overline{\mathbb{Q}}_\ell[\check{\Lambda}^+] = \overline{\mathbb{Q}}_\ell[\check{\Lambda}]^W.$$