Preliminary Examination: Electricity and Magnetism **May 14, 2005**

Written by G. Domokos

This is a closed book, closed notes exam. Some useful formulæ are provided. Please, be sure that you can be properly identified: print your name on every sheet of paper you hand in. Start every problem on a new sheet of paper and show all relevant work. (However, discard sheets on which you tried something, but it did not lead to anything useful.) Please, make sure that you explain what you are doing. The time available for this exam is 3 hours. The exam consists of 5 problems: each one carries a weight of 20 points, to a maximum of 100 points.

Good luck!

1. A very long wire of diameter a is held at a constant potential. The charge density per unit length on the wire is λ.

   a) Determine the potential and the electric field as a function of the distance from the wire.

   b) A second wire carrying a charge per unit length $-\lambda$ and of the same diameter is now placed at a distance d parallel to the first wire. Assuming $d \gg a$, determine the potential difference between the wires.

2. A short wave radio receiver receives simultaneously two signals from a transmitter 500km away, one along the surface of the Earth, and one by reflection from a portion of the ionosphere situated at a height of 200km. The layer in the ionosphere acts as a perfect horizontal mirror for the waves. The frequency of the transmitted wave is 10MHz. One observes that the combined signal strength varies from maximum to minimum at 8 times per minute.

   Determine the vertical speed of the reflecting ionosphere layer. (Assume that the Earth is flat and, if necessary, that the reflecting layer of the ionosphere is moving slowly.)

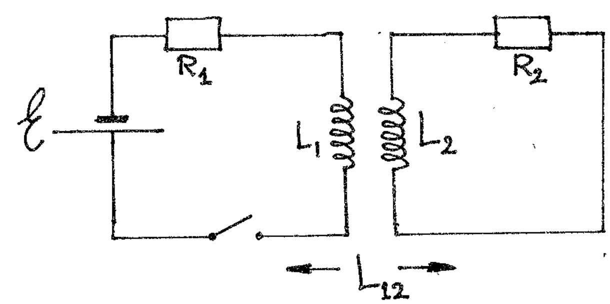
3. A long coaxial cable of length l consists of an inner conductor of radius a and an outer conductor (the mantle) of radius b. The coaxial cable is connected to a battery at one end and to a resistor at the other. The

inner conductor carries a uniform charge per unit length λ and a steady current I. The outer conductor has the opposite charge and current.

Determine the electromagnetic momentum stored in the fields.

4. A member of a colony on a moon of Jupiter is required to salute the UN flag at the same time as it is being done on Earth, at noon in New York. If observers in all inertial frames are to agree that she has performed her duty, for how long must she salute? (The distance between Earth and Jupiter is, approximately, 8×10^8km. You may ignore the relative motion of the Earth and the moon of Jupiter.) Consider only observers moving along the line joining the Earth and the moon of Jupiter at a uniform speed.

5. Consider the circuit shown on the following Figure. The EMF of the



battery is $\mathcal{E}$ and it has a negligible internal resistance. Assuming that the switch is closed at time $t = 0$, determine the current in the secondary circuit as a function of time.

[illegible] ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

In matter:

$$\nabla \cdot \vec{D} = \rho_f$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Auxiliary Fields

Definitions:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

Linear media:

$$\vec{P} = \epsilon \chi \vec{E} \qquad \vec{D} = \epsilon \vec{E}$$

$$\vec{M} = \chi_m \vec{H} \qquad \vec{H} = \frac{1}{\mu}\vec{B}$$

Potentials

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\vec{P} = \epsilon_0 \int (\vec{E} \times \vec{B}) d\tau$

Poynting vector: $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c} q^2 a^2$

Electric dipole

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{r}}{r^2}$$

$$\mathbf{E}_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}]$$

For a dielectric with polarization P, the potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int\frac{-\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' + \frac{1}{4\pi\varepsilon_0}\int\frac{\mathbf{P}(\mathbf{r}')\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}da'$$

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

Quadrupole moment tensor : $Q_{ij} = \int(3\,x_i x_j - r^2\delta_{ij})\,\rho(r$

Current density : $\mathbf{J} = n\,q\,\mathbf{v} = \rho\,\mathbf{v}$

Ohm's law : $\mathbf{J} = \sigma\,\mathbf{E}$

Capacitor : $Q = CV$

Continuity Equation : $\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0,$

Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{I\,d\boldsymbol{\ell}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau'$$

Vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau'$$

Axial field due to a current loop of radius a and current I

$$B = \frac{\mu_0\,I\,(\pi a^2)}{2\pi\,(a^2+z^2)^{3/2}}$$

Magnetic dipole

$$\mathbf{A}_{dipole} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^2}$$

$$\mathbf{B}_{dipole} = \frac{\mu_0}{4\pi} \frac{m}{r^3} [2\cos\theta \, \hat{\mathbf{r}} + \sin\theta \, \hat{\theta}]$$

For a magnetized object

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \mathbf{n}_1}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$$

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{-\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{1}{4\pi} \int \frac{\mathbf{n}_1 \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

Lorentz condition : $\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0$

Plane light wave propagating in $\mathbf{n}_1$ (unit vector) direction

$$\mathbf{B} = \frac{1}{v} \mathbf{n}_1 \times \mathbf{E} = \frac{k}{\omega} (\mathbf{n}_1 \times \mathbf{E}), \qquad v^2 = \frac{1}{\mu\varepsilon}$$

index of refraction $\quad n = \frac{c}{v}$

Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Time-average power density $\quad P = |\langle S \rangle| = \frac{1}{2} \mathrm{Re}(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2} v \varepsilon |E_0|^2$

Some useful mathematical information:

1. General solution of $\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) = 0$ in spherical coordinates

$$V(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + \frac{B_n}{r^{n+1}})\,P_n(\cos\theta)$$

where $P_n(\cos\theta)$ is the Legendre polynomials and

$$\int_{-\pi}^{\pi} P_m(\cos\theta)\,P_n(\cos\theta)\,d(\cos\theta) = \frac{2}{2n+1}\,\delta_{mn}$$

$P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$, $P_3(\cos\theta) = (5\cos^3\theta - 3\cos\theta)/2$.

2. General solution of $\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho}) + \frac{1}{\rho}\frac{\partial^2 V}{\partial\phi^2} = 0$ in cylindrical coordinates

$$V(\rho,\phi) = D_0 + A_0\ln\rho + \sum_{\nu}(A_\nu\rho^\nu + \frac{B_\nu}{\rho^\nu})\,(C_\nu\sin\nu\phi + D_\nu\cos\nu\phi)$$

For unrestricted ϕ, ν = positive non-zero integer.

$$\int_{-\pi}^{\pi}\sin m\phi\,\sin n\phi\,d\phi = \pi\,\delta_{mn},\quad \int_{-\pi}^{\pi}\cos m\phi\,\cos n\phi\,d\phi = \pi\,\delta_{mn},\quad \int_{-\pi}^{\pi}\sin m\phi\,\cos n\phi\,d\phi = 0$$

3. General solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, $\quad V(x,y) = \sum_{a_n}\{\exp(\pm a_n x)\}\,\{\exp(\pm i a_n y)\}$

$e^{\pm ix} = \cos x \pm i\sin x$, $\quad e^{\pm x} = \cosh x \pm \sinh x$

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn},\quad \int_0^a \cos\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn},\quad \int_0^a \sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = 0,$$

4. Binomial Theorem : $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$

5. Some frequently used integrals :

$$\int\frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}},\quad \int\frac{dx}{(a^2+x^2)^{1/2}} = \ln(x+\sqrt{a^2+x^2})$$

$$\int[\sqrt{a^2+x^2}\,]^{\pm n}\,x\,dx = \frac{1}{2\pm n}[\sqrt{a^2+x^2}\,]^{2\pm n}\quad (n \neq 2 \text{ or } -2)$$

$$\int\frac{x^2dx}{(a^2+x^2)^{3/2}} = \frac{-x}{\sqrt{a^2+x^2}} + \ln(x+\sqrt{a^2+x^2})$$

6. Levi-Civita Tensors : $\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$, $\qquad \varepsilon_{ijk}\,\varepsilon_{ijn} = 2\,\delta_{kn}$

$$(\mathbf{A}\times\mathbf{B})_i = \varepsilon_{ijk}\,A_j\,B_k$$