Electricity and Magnetism Preliminary Exam
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2004 May 13th written by David Neufeld

Attempt all five problems, using the S.I. (MKS) system of units. Partial credit will be given for partially-correct solutions, so do as much as you can on each problem. Please make your work neat and readable. The exam is closed book, no notes (except for the attached sheet of possibly-useful expressions). All questions carry equal credit.

1. Two spherical cavities, of radii a and b, are hollowed out from the interior of a neutral conducting sphere of radius R. At the center of each cavity a point charge is placed – call these charges q_a and q_b.

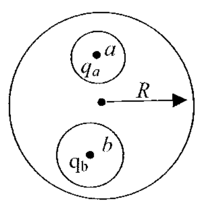
(a) What are the charge densities σ_a, σ_b and σ_R on the surfaces of the two cavities and of the conducting sphere?

(b) What is the field outside the conductor?

(c) What is the field within each cavity?

(d) What are the electrostatic forces on q_a and q_b?

(e) Which of these answers would change if a third charge, q_c, were brought near the conductor?



2. A capacitor is constructed from two circular parallel plates of radius R.

(a) The plates are separated by a distance $d_1 \ll R$ and connected to a battery which applies a potential difference V_1. What is the charge density on the plates, the attractive force between the plates and the energy stored in the electric field.

(b) The battery is now disconnected and the plates are pulled apart to separation d_2 ($> d_1$ but still $\ll R$). Now what is the potential difference between the plates and the energy stored in the electric field?

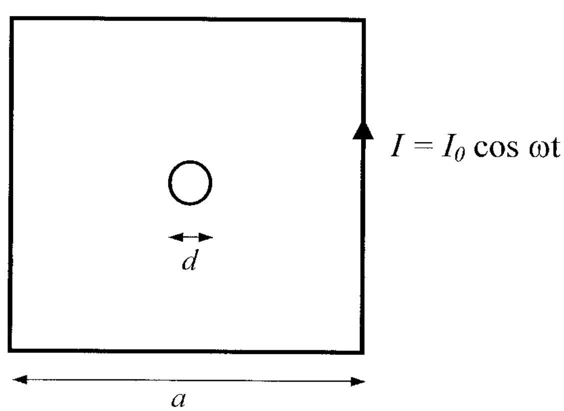
(c) Show explicitly that the increase in the stored energy is equal to the work done in separating the plates.

3. A small circular loop of diameter d is placed at the center of a wire square of side $a \gg r$ that carries an alternating current $I = I_0 \cos \omega t$. The loop and the square lie in the same plane.

Assuming that any induced current in the circular loop produces a negligible magnetic field, compute the magnetic field at the center of the square.

If the resistance of the loop is R, compute the induced current in the loop as a function of time. (You may assume that R is large enough and/or d small enough that the self–inductance of the circular loop may be neglected.

What is the time averaged rate of Ohmic heating in the loop? Where does the required energy come from?



4. A plane, monochromatic electromagnetic wave propagates in an infinite medium of conductivity σ. The medium has negligible magnetic or electric polarizability, i.e. $\varepsilon = \varepsilon_0$, $\mu = \mu_0$.

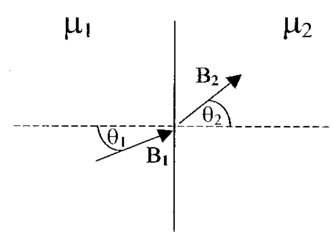
Starting with Maxwell's equations and Ohm's Law

(a) derive a wave equation obeyed by the electric field

(b) derive a dispersion relation relating the wavenumber, k, to the angular frequency, ω

(c) show that the amplitude of the wave decreases exponentially in the direction of propagation, and obtain an expression for the attenuation length scale in the limit of poor conductivity, $\sigma \ll \omega\varepsilon_0$.

5. At the interface between one linear magnetic material and another, the magnetic field lines kink as shown below. Show that the perpendicular component of **H** and the parallel component of **B** are each continuous, and thereby derive a relationship between the angles θ_1 and θ_2 and the permeabilities μ_1 and μ_2. (Assume that there are no free currents flowing within the material.)



Useful Equations

- Maxwell's Equations:

$$\begin{array}{ll} \nabla \cdot \vec{D} = \rho(\vec{x}) & \oint \vec{D} \cdot d\vec{a} = Q_{enclosed} \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & \oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial \Phi_B}{\partial t} \\ \nabla \cdot \vec{B} = 0 & \oint \vec{B} \cdot d\vec{a} = 0 \\ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} & \oint \vec{H} \cdot d\vec{\ell} = I_{enclosed} + \frac{\partial \Phi_D}{\partial t} \end{array}$$

- Boundary Conditions: $\hat{n}$ is normal to the plane boundary, σ and $\vec{K}$ are surface charge and current densities.

$$\begin{array}{ll} \hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \sigma & \hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \\ \hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0 & \hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K} \end{array}$$

- Static Potentials: φ and $\vec{A}$ are the electric scalar and magnetic vector potentials.

$$\vec{E} = -\nabla\varphi \quad \vec{H} = \nabla \times \vec{A}$$

- Potential of a dipole:

$$\varphi(\vec{r}) = \frac{\vec{P} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$$

- Dipole Moment:

$$\vec{P} = \int \vec{x}\rho(\vec{x})d^3x$$

- Lorentz Force:

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v} \times \vec{B}$$

- Homogeneous Media:

$$\begin{array}{l} \vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P} \\ \vec{B} = \mu\vec{H} = \mu_0\left(\vec{H} + \vec{M}\right) \end{array}$$

- Definitions:
  - Capacitance - $C = Q/V$
  - Resistance - $V = IR$
  - Inductance - $V = -LdI/dt$

Electric dipole

$$V_{dipole} = \frac{1}{4\pi\varepsilon_o} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$$

$$\mathbf{E}_{dipole} = \frac{1}{4\pi\varepsilon_o} \frac{p}{r^3} [2\cos\theta\ \hat{\mathbf{r}}_1 + \sin\theta\ \hat{\theta}_1]$$

For a dielectric with polarization $\mathbf{P}$, the potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'$$

$$= \frac{1}{4\pi\varepsilon_o} \int \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' + \frac{1}{4\pi\varepsilon_o} \int \frac{\mathbf{P}(\mathbf{r}') \cdot \mathbf{n}_1}{|\mathbf{r} - \mathbf{r}'|} da'$$

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P}$$

Quadrupole moment tensor : $Q_{ij} = \int (3\, x_i x_j - r^2 \delta_{ij})\, \rho(\mathbf{r})\, d\tau$

Current density : $\mathbf{J} = n\, q\, \mathbf{v} = \rho\, \mathbf{v}$

Ohm's law : $\mathbf{J} = \sigma\, \mathbf{E}$

Capacitor : $Q = CV$

Continuity Equation : $\nabla \cdot \mathbf{J} + \frac{\partial \rho_f}{\partial t} = 0,$

Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int \frac{I\, d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_o}{4\pi} \int \frac{\mathbf{J}_f(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'$$

Vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int \frac{\mathbf{J}_f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau'$$

$$d\vec{A}(r) = \frac{\mu_0}{4\pi} \frac{d\vec{I}}{r}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V$$

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_{\text{volume}} (\nabla \cdot \mathbf{A})\, d\tau = \oint_{\text{surface}} \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int_{\text{surface}} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_{\text{line}} \mathbf{A} \cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}$ coul²/N-m²	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}$ N/A²	(permeability of free space)
$c = 3.00 \times 10^8$ m/sec	(speed of light)
$e = 1.60 \times 10^{-19}$ coul	(charge of the electron)
$m = 9.11 \times 10^{-31}$ kg	(mass of the electron)

CONVERSION FROM SPHERICAL TO CARTESIAN COORDINATES

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{i} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{j} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ \hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} \\ \hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \end{cases}$$

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k};\; d\tau = dx\,dy\,dz$

Gradient. $\nabla t = \dfrac{\partial t}{\partial x}\hat{i} + \dfrac{\partial t}{\partial y}\hat{j} + \dfrac{\partial t}{\partial z}\hat{k}$

Divergence. $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$

Curl. $\nabla \times \mathbf{v} = \left(\dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z}\right)\hat{i} + \left(\dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x}\right)\hat{j} + \left(\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y}\right)\hat{k}$

Laplacian. $\nabla^2 t = \dfrac{\partial^2 t}{\partial x^2} + \dfrac{\partial^2 t}{\partial y^2} + \dfrac{\partial^2 t}{\partial z^2}$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi};\; d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $\nabla t = \dfrac{\partial t}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial t}{\partial \theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial t}{\partial \phi}\hat{\phi}$

Divergence. $\nabla \cdot \mathbf{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 v_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi}$

Curl.
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian. $\nabla^2 t = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial t}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial t}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 t}{\partial \phi^2}$

CYLINDRICAL $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z};\; d\tau = r\,dr\,d\phi\,dz$

Gradient. $\nabla t = \dfrac{\partial t}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial t}{\partial \phi}\hat{\phi} + \dfrac{\partial t}{\partial z}\hat{z}$

Divergence. $\nabla \cdot \mathbf{v} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(r v_r) + \dfrac{1}{r}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{\partial v_z}{\partial z}$

Curl.
$$\nabla \times \mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$$

Laplacian. $\nabla^2 t = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial t}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 t}{\partial \phi^2} + \dfrac{\partial^2 t}{\partial z^2}$