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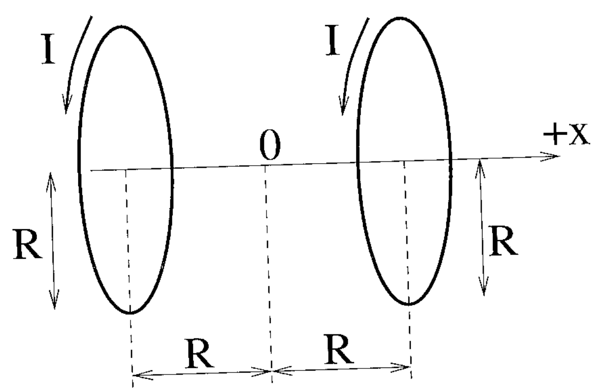
Preliminary Exam: Electricity and Magnetism

This is a closed book exam. Page 6 is a sheet of useful equations. Everything that you need is derivable from them in at most a few steps. Please attempt all five problems. Good luck!

1. [15 points] Here in Bloomberg, in our Advanced Lab course we use Helmholtz coils to provide magnetic field which is almost homogeneous in a small region of space. For the sake of simplicity, the Helmholtz coils can be represented by two identical rings, each of radius R with current I. The rings are coaxial and placed $2R$ apart, and the current is flowing in the same sense in both rings.

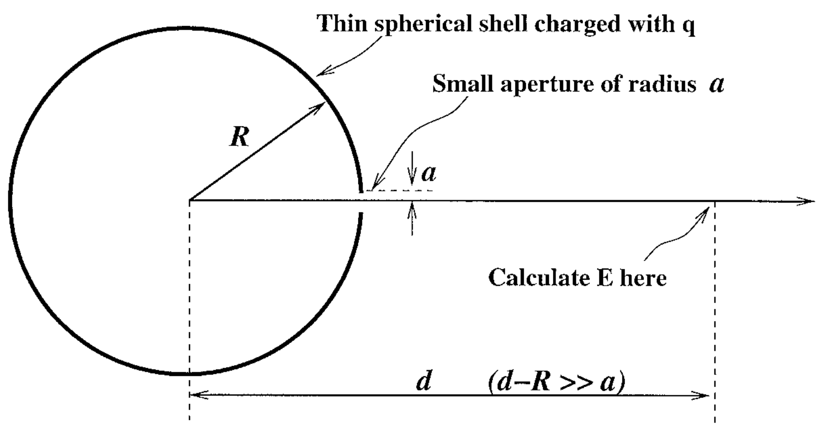
What is the strength and the direction of the magnetic field: a) at the mid-point between the rings? b) at the distance x from the mid-point along the axis of both coils.



0 = mid–point between coils and the origin of the x–axis

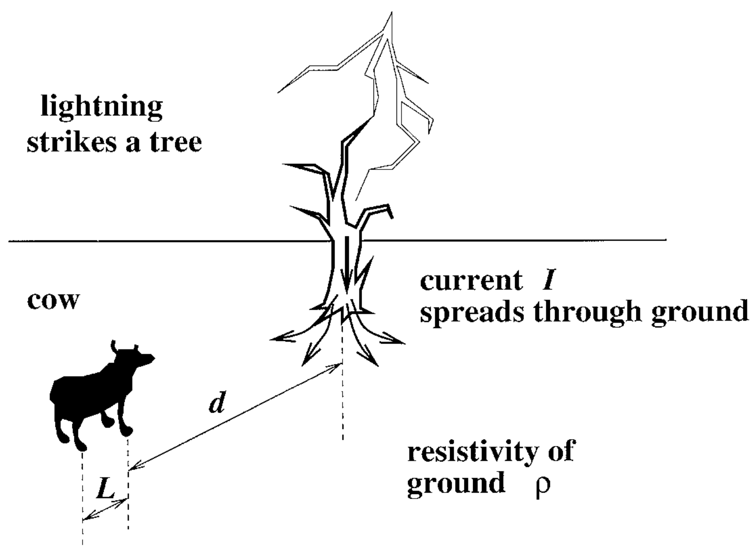
2. [15 points] A thin dielectric spherical shell of radius R is suspended in vacuum. The shell is uniformly charged with a total positive charge of q. A small hole of radius a is drilled in the shell as indicated in the figure.

What is the electric field along the axis of the hole, at a distance d from the center of the shell, where $d - R >> a$. Hint: would the answer change if, instead of drilling a hole, a small disk of radius a charged negatively (but with the same surface charge density) was placed just in front of the shell, at the distance $h << a$ from its surface?



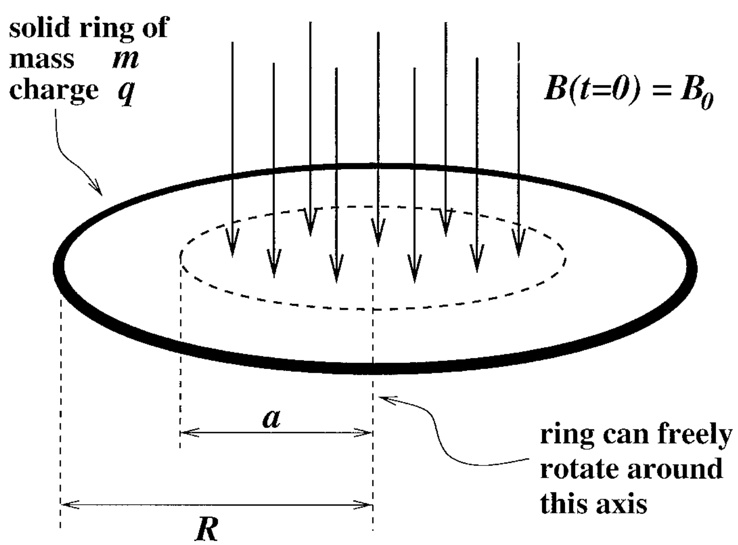
3. [20 points] In Illinois (assumed to be perfectly horizontal and flat) a cow stands near a tree. A bolt of lightning strikes the tree tree, and the current I spreads from the tree's roots isotropically through the ground. The ground can be considered a uniform and isotropic resistive medium with resistivity ρ. The cow's length (from front to hind legs) is L, and it can sustain a voltage $V_{\max}$.

What is the cow's safe distance d from the tree – the minimum distance from the tree at which the cow will not be electrocuted regardless of its orientation with respect to the tree?



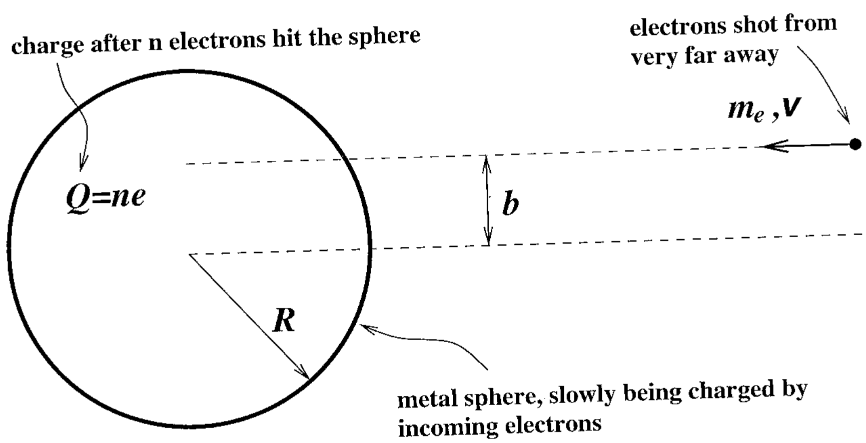
4. [25 points] A dielectric ring of radius R and mass m is charged uniformly around its circumference with a charge q. The ring is in a horizontal plane and can rotate around its (vertical) axis without friction. In the circular area $a < R$ inside the ring there is a homogeneous magnetic field oriented in the vertical direction. The whole system is in vacuum.

Initially, the ring is at rest ($\omega(t=0)=0$), and the strength of the magnetic field is $B(t=0)=B_0$. At $t=0$, the strength of the magnetic field starts to linearly decrease with time, and eventually reaches zero. What is the angular velocity of the ring at the moment when the magnetic field disappears? Would the answer change if B is decreased to zero non-linearly? Assume that the magnetic field generated by the rotating ring is much smaller than B_0.



5. [25 points] A metal sphere of radius R is suspended in vacuum and fixed. Electrons are being shot, one at a time, with the initial velocity v ($v << c$) towards the sphere so that the sphere is slowly being charged. Mass of the electron is m_e and its charge is e.

What is the final charge of the sphere, Q, if the initial velocity of each electron is parallel to the axis of the sphere, but offset by b ($0 < b < R$), as indicated on the figure. (Assume that the final charge $Q >> e$ and thus the effect of the image charges can be neglected.)



Useful Equations

- Maxwell's Equations:

$$
\begin{array}{ll}
\nabla \cdot \vec{D} = \rho(\vec{x}) & \oint \vec{D} \cdot d\vec{a} = Q_{enclosed} \\
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & \oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial \Phi_B}{\partial t} \\
\nabla \cdot \vec{B} = 0 & \oint \vec{B} \cdot d\vec{a} = 0 \\
\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} & \oint \vec{H} \cdot d\vec{\ell} = I_{enclosed} + \frac{\partial \Phi_D}{\partial t}
\end{array}
$$

- Boundary Conditions: $\hat{n}$ is normal to the plane boundary, σ and $\vec{K}$ are surface charge and current densities.

$$
\begin{array}{ll}
\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \sigma & \hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \\
\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0 & \hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}
\end{array}
$$

- Static Potentials: φ and $\vec{A}$ are the electric scalar and magnetic vector potentials,

$$\vec{E} = -\nabla\varphi \quad \vec{H} = \nabla \times \vec{A}$$

- Potential of a dipole:

$$\varphi(\vec{r}) = \frac{\vec{P} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$$

- Dipole Moment:

$$\vec{P} = \int \vec{x}\rho(\vec{x}) d^3x$$

- Lorentz Force:

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v} \times \vec{B}$$

- Law of Biot-Savard:

$$\vec{B}(\vec{r}) = \int d\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{\ell} \times \vec{r}}{r^3}$$

- Homogeneous Media:

$$
\begin{array}{l}
\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P} \\
\vec{B} = \mu\vec{H} = \mu_0\left(\vec{H} + \vec{M}\right)
\end{array}
$$

- Definitions:
  - Capacitance - $C = Q/V$
  - Resistance - $V = IR$
  - Inductance - $V = -LdI/dt$

- Lorentz transformations: for a boost along coordinate x^1:

$$\begin{aligned}\bar{x}^0 &= \gamma(x^0 - \beta x^1)\\ \bar{x}^1 &= \gamma(x^1 - \beta x^0)\\ \bar{x}^2 &= x^2\\ \bar{x}^3 &= x^3\end{aligned}$$

- Lorentz transformations of $\vec{E}$ and $\vec{B}$ along the x-axis:

$$\bar{E}_x = E_x \quad \bar{E}_y = \gamma(E_y - vB_z) \quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma(B_y + \tfrac{v}{c^2}E_z) \quad \bar{B}_z = \gamma(B_z - \tfrac{v}{c^2}E_y)$$

- Field tensor and its dual tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

- Harmonic oscillator: the equation of motion

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

results in harmonic oscillations with the angular frequency ω. Period of such oscillations is

$$T = \frac{2\pi}{\omega}$$