Preliminary Exam on Electricity and Magnetism (Spring 2001)

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Do all problems, showing all your work explicitely. Do not get too hung up on any one problem, and do not spend all your time on that one, thereby neglecting the rest of the exam. Concentrate your attention on setting up the problem. Good luck!

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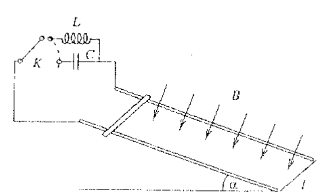
1. (15pts) A metal sphere of radius R carries a total charge Q. What is the force of repulsion between the Northern and Southern hemispheres. (Hint: consider cutting the sphere into two hemispheres, and moving them slightly apart).

2. (20pts) There is a charge $+q$ at a position $(a,a,0)$. There are two perfectly conducting planes in the xz and yz plane.
   a. What is the force acting on the charge?
   b. How much work does it take to move the charge to infinity?
   c. What is the radial dependence of the electrostatic potential at $r>>a$?

3. (20pts) Pulsars are thought to be rotating neutron stars, with a typical radius of 10km, a rotational period of 10^{-3} s, and a surface magnetic field of 10^{8} T. using dimensional arguments, determine an order of magnitude estimate for
   a. the magnetic dipole moment of the star
   b. the expected radiated power

4. (25pts) A frictionless metal bar of mass m is sliding down on two perfectly conducting rails separated by distance l, on top of an inclined plane with angle α with respect to the ground. There is a homogeneous magnetic field B, perpendicular to the plane. We close the circuit formed by the bar and the two rails with (a) a capacitor C, (b) an inductance L. Describe the motion of the bar in the two cases!



5. (20pts) Show that the effects of two subsequent Lorentz transformations with velocities u and v, along the same axis, can be described also with a single Lorentz transformation. What is the velocity V of the combined transformation?

Some Physics Formulae:

Electric dipole

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$$

$$\mathbf{E}_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{p}{r^3} [2\cos\theta\, \hat{\mathbf{r}}_1 + \sin\theta\, \hat{\boldsymbol{\theta}}_1]$$

For a dielectric with polarization **P**, the potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0} \int \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' + \frac{1}{4\pi\varepsilon_0} \int \frac{\mathbf{P}(\mathbf{r}') \cdot \mathbf{n}_1}{|\mathbf{r} - \mathbf{r}'|} da'$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

Quadrupole moment tensor : $Q_{ij} = \int (3\, x_i x_j - r^2 \delta_{ij})\, \rho(\mathbf{r})\, d\tau$

Current density : $\mathbf{J} = n\, q\, \mathbf{v} = \rho\, \mathbf{v}$

Ohm's law : $\mathbf{J} = \sigma\, \mathbf{E}$

Capacitor : $Q = CV$

Continuity Equation : $\nabla \cdot \mathbf{J} + \frac{\partial \rho_f}{\partial t} = 0,$

Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_f(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'$$

Vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau'$$

Axial field due to a current loop of radius a and current I

$$B = \frac{\mu_0\, I\, (\pi a^2)}{2\pi\, (a^2 + z^2)^{3/2}}$$

Magnetic dipole

$$\mathbf{A}_{\text{dipole}} = \frac{\mu_o}{4\pi}\ \frac{\mathbf{m} \times \mathbf{r}}{r^3}$$

$$\mathbf{B}_{\text{dipole}} = \frac{\mu_o}{4\pi}\frac{m}{r^3}\left[2\cos\theta\ \hat{\mathbf{r}}_1 + \sin\theta\ \hat{\boldsymbol{\theta}}_1\right]$$

For a magnetized object

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{\mu_o}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}') \times \mathbf{n}_1}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M})$$

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi}\int \frac{-\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{1}{4\pi}\int \frac{\mathbf{n}_1 \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

Lorentz condition : $\nabla \cdot \mathbf{A} + \mu\varepsilon\frac{\partial\phi}{\partial t} = 0$

Plane light wave propagating in $\mathbf{n}_1$ (unit vector) direction

$$\mathbf{B} = \frac{1}{v}\mathbf{n}_1 \times \mathbf{E} = \frac{k}{\omega}(\mathbf{n}_1 \times \mathbf{E}), \qquad v^2 = \frac{1}{\mu\varepsilon},$$

index of refraction $\quad n = \frac{c}{v}$

Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Time-average power density $\quad P = |\langle S \rangle| = \frac{1}{2}\mathrm{Re}\,|\mathbf{E} \times \mathbf{H}^*| = \frac{1}{2}v\,\varepsilon\,|E_o|^2$

Maxwell stress tensor: $T_{ij} = -D_iE_j - B_iH_j + \frac{1}{2}\delta_{ij}(\vec{B}\cdot\vec{H} + \vec{D}\cdot\vec{E})$.

$$f_j = \partial_k\left[D_kE_j + B_kH_j - \tfrac{1}{2}\delta_{jk}(\vec{B}\cdot\vec{H} + \vec{D}\cdot\vec{E})\right] = -\partial_k \overleftrightarrow{T}_{kj}$$

Resonant Cavities

$$\vec{H}_{0,t} = \frac{1}{\varepsilon\mu\omega^2 - k^2}\left(\vec{\nabla}_t\frac{\partial H_{0,z}}{\partial z} + i\omega\varepsilon\hat{k}\times\vec{\nabla}_t E_{0,z}\right)$$

$$\vec{E}_{0,t} = \frac{1}{\varepsilon\mu\omega^2 - k^2}\left(\vec{\nabla}_t\frac{\partial E_{0,z}}{\partial z} - i\mu\omega\hat{k}\times\vec{\nabla}_t H_{0,z}\right)$$

Circuits.
Ohm's law for circuits: $V = RI$.
Parallel capacitors: $C = C_1 + C_2$
Capacitors in series: $1/C = 1/C_1 + 1/C_2$
Parallel resistors: $1/R = 1/R_1 + 1/R_2$
Resistors in series: $R = R_1 + R_2$
Relativistic kinematics of point particles.
Energy: $E = (\mathbf{p}^2c^2 + m^2c^4)^{1/2}$
Momentum: $\mathbf{p} = m\mathbf{v}/(1 - v^2/c^2)^{1/2}$
Angular momentum: $\mathbf{J} = \mathbf{x} \times \mathbf{p}$
Radiating particle.
Radiation field of a non relativistic particle ($v \ll c$):

$$\mathbf{B} = \frac{e}{4\pi\epsilon_0 c^2 r^2}\dot{\mathbf{v}} \times \mathbf{r}$$

$$\mathbf{E} = \frac{\mathbf{B} \times \mathbf{r}}{r}$$

Energy loss due to radiation ($v \ll c$):

$$\frac{dE}{dt} = -\frac{e^2\dot{\mathbf{v}}^2}{4\pi\epsilon_0 c^3}$$

Radiation damping force:

$$\mathbf{F} = \frac{e^2}{6\pi\epsilon_0 c^3}\frac{d^2\mathbf{v}}{dt^2}$$

The total radiated power for electric and magnetic dipoles is

$$\langle P\rangle = \frac{\mu_0 p_0^2\omega^4}{12\pi c} \qquad \langle P\rangle = \frac{\mu_0 m_0^2\omega^4}{12\pi c^3}$$

The Lorentz-transformation is

(i) $\bar{x} = \gamma(x - vt)$.

(ii) $\bar{y} = y$.

(iii) $\bar{z} = z$.

(iv) $\bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$.

Some useful mathematical information:

1. General solution of $\frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial V}{\partial\theta} = 0$ in spherical coordinates

$$V(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + \frac{B_n}{r^{n+1}}) P_n(\cos\theta)$$

where $P_n(\cos\theta)$ is the Legendre polynomials and

$$\int_{-\pi}^{\pi} P_m(\cos\theta)\, P_n(\cos\theta)\, d(\cos\theta) = \frac{2}{2n+1}\delta_{mn}$$

$P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = (3\cos^2\theta-1)/2$, $P_3(\cos\theta) = (5\cos^3\theta-3\cos\theta)/2$,

2. General solution of $\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho}) + \frac{1}{\rho}\frac{\partial^2 V}{\partial\phi^2} = 0$ in cylindrical coordinates

$$V(\rho,\phi) = D_o + A_o\, ln\rho + \sum_{\nu}(A_\nu \rho^\nu + \frac{B_\nu}{\rho^\nu})\{C_\nu \sin\nu\phi + D_\nu \cos\nu\phi\}$$

For unrestricted ϕ, ν = positive non-zero integer.

$$\int_{-\pi}^{\pi} \sin m\phi\, \sin n\phi\, d\phi = \pi\,\delta_{mn}, \quad \int_{-\pi}^{\pi} \cos m\phi\, \cos n\phi\, d\phi = \pi\,\delta_{mn}, \quad \int_{-\pi}^{\pi} \sin m\phi\, \cos n\phi\, d\phi = 0$$

3. General solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, $\quad V(x,y) = \sum_{a_n}\{\exp(\pm a_n x)\}\{\exp(\pm i a_n y)\}$

$$e^{\pm ix} = \cos x \pm i \sin x, \quad e^{\pm x} = \cosh x \pm \sinh x$$

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn}, \quad \int_0^a \cos\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn}, \quad \int_0^a \sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = 0,$$

4. Binomial Theorem : $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + ...$

5. Some frequently used integrals :

$$\int\frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}, \quad \int\frac{dx}{(a^2+x^2)^{1/2}} = ln(x + \sqrt{a^2+x^2})$$

$$\int[\sqrt{a^2+x^2}\,]^{\pm n}\, x dx = \frac{1}{2\pm n}[\sqrt{a^2+x^2}\,]^{2\pm n} \quad (n \neq 2 \text{ or } -2)$$

$$\int\frac{x^2 dx}{(a^2+x^2)^{3/2}} = \frac{-x}{\sqrt{a^2+x^2}} + ln(x + \sqrt{a^2+x^2})$$

6. Levi-Civita Tensors : $\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$, $\qquad \varepsilon_{ijk}\,\varepsilon_{ijn} = 2\,\delta_{kn}$

$$(\mathbb{A} \times \mathbb{B})_i = \varepsilon_{ijk}\, A_j\, B_k,$$

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_{\text{volume}} (\nabla \cdot \mathbf{A})\, d\tau = \oint_{\text{surface}} \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int_{\text{surface}} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_{\text{line}} \mathbf{A} \cdot d\mathbf{l}$

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k};\ d\tau = dx\,dy\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial x}\hat{i} + \frac{\partial t}{\partial y}\hat{j} + \frac{\partial t}{\partial z}\hat{k}$$

Divergence. $$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

Laplacian. $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi};\ d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\phi}$$

Divergence. $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl. $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

Laplacian. $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

CYLINDRICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z};\ d\tau = r\,dr\,d\phi\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial\phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

Divergence. $$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla\times\mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial\phi}\right]\hat{z}$$

Laplacian. $$\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$