

**Preliminary Exam on Electricity and Magnetism (spring 2000)**

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Do all problems. The time limit is 3 hours. Please start each problem on a new sheet and show all work. If you use more than one blue book, sign all books and indicate the problem numbers on the front page.

There are 1 sheet of exam problems, 2 sheets of formulae and 4 sheets of math formulae. SI units are preferred but not required. If you cannot solve a problem completely, demonstrate that you are on the right track.

- (10 pts) Consider a long solenoid of radius  $a$  with  $n$  turns/unit length of wires wrapped around its circumference. Suppose the current ( $I$ ) in the wires is increasing linearly with time, (i.e.,  $dI/dt = K = \text{constant}$ ). Find the electric field  $\mathbf{E}$  inside and outside the solenoid.
- (15 pts) Two concentric spherical shells have inner and outer radii of  $a$  and  $b$  respectively. At  $r = a$ , the potential is  $V = V_0 \cos\theta$  where  $V_0$  is a constant, and  $\theta$  is the angle with respect to an axis through the center. At  $r = b$ , the potential  $V = 0$ . Find the potential in the space between the shells ( $a \leq r \leq b$ ).
- (15 pts) Two point charges  $q$  and  $-3q$  are separated by a distance  $S$ . Find the equipotential surface with zero potential.
- (20 pts) We wish to find out the fraction of light intensity transmitted at normal incidence through glass for those who wear glasses. We assume that the light is linearly polarized, monochromatic with a wavelength  $\lambda$  and intensity  $W$ . Light transmission through air (index of refraction  $n_1 = 1$ ) and glass (index of refraction  $n_2 = 1.3$ ) are assumed to be without loss. The magnetic permeability of air and glass can be taken to be 1.
  - What should be appropriate boundary conditions at the air/glass interface?
  - Find the expressions for the  $\mathbf{E}$  part and  $\mathbf{B}$  (or  $\mathbf{H}$ ) part of the electromagnetic wave in air and those in glass.
  - What is the fraction of light that transmits through the glasses and enters the eyes?
- (20 pts) An infinitely long thin wire carrying a DC current  $I$  is surrounded coaxially by a cylindrical shell of inner radius  $a$  and outer radius  $b$ , of a linear magnetic material of constant susceptibility  $\chi$ .
  - Find  $\mathbf{B}$ ,  $\mathbf{H}$  and  $\mathbf{M}$  everywhere
  - Find the bound current density (or magnetization current density), both volume current density ( $\nabla \times \mathbf{M}$ ) and surface current ( $\mathbf{M} \times \mathbf{n}_1$ ).
- (20 pts) A conducting sphere of radius  $R$  carries a total charge of  $Q$ , which resides on the surface. The sphere is spinning about a diameter with an angular velocity  $\omega$ .
  - Find the magnetic induction ( $\mathbf{B}$  field) at the center of the sphere.
  - Find the effective magnetic moment and its direction for the spinning charged sphere. At large distances from the sphere ( $r \gg R$ ), what will be a reasonable form for the
  - magnetic induction ( $\mathbf{B}$ )
  - the vector potential ( $\mathbf{A}$ ), where  $\mathbf{B} = \nabla \times \mathbf{A}$ ?

### Some Physics Formulae:

Electric dipole

$$V_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$$

$$\mathbf{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{p}{r^3} [2\cos\theta \hat{\mathbf{r}}_1 + \sin\theta \hat{\boldsymbol{\theta}}_1]$$

For a dielectric with polarization  $\mathbf{P}$ , the potential is

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}' \\ &= \frac{1}{4\pi\epsilon_0} \int \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' + \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{P}(\mathbf{r}') \cdot \mathbf{n}_1 da'}{|\mathbf{r} - \mathbf{r}'|} \end{aligned}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

Quadrupole moment tensor :  $Q_{ij} = \int (3 x_i x_j - r^2 \delta_{ij}) \rho(\mathbf{r}) d\mathbf{r}$

Current density :  $\mathbf{J} = n q \mathbf{v} = \rho \mathbf{v}$

Ohm's law :  $\mathbf{J} = \sigma \mathbf{E}$

Capacitor :  $Q = CV$

Continuity Equation :  $\nabla \cdot \mathbf{J} + \frac{\partial \rho_f}{\partial t} = 0$

Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}'$$

Vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'$$

Axial field due to a current loop of radius  $a$  and current  $I$

$$B = \frac{\mu_0 I (\pi a^2)}{2\pi (a^2 + z^2)^{3/2}}$$

Magnetic dipole

$$\mathbf{A}_{\text{dipole}} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}$$

$$\mathbf{B}_{\text{dipole}} = \frac{\mu_0 m}{4\pi r^3} [2\cos\theta \hat{\mathbf{r}}_1 + \sin\theta \hat{\boldsymbol{\theta}}_1]$$

For a magnetized object

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \mathbf{n}_1}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$$

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{-\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{1}{4\pi} \int \frac{\mathbf{n}_1 \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

$$\text{Lorentz condition: } \nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t} = 0$$

Plane light wave propagating in  $\mathbf{n}_1$  (unit vector) direction

$$\mathbf{B} = \frac{1}{v} \mathbf{n}_1 \times \mathbf{E} = \frac{k}{\omega} (\mathbf{n}_1 \times \mathbf{E}), \quad v^2 = \frac{1}{\mu\epsilon}$$

$$\text{index of refraction } n = \frac{c}{v}$$

$$\text{Poynting vector } \mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\text{Time-average power density } P = \langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H}^* \} = \frac{1}{2} v \epsilon |\mathbf{E}_0|^2$$

$$\text{Maxwell stress tensor: } T_{ij} = -D_i E_j - B_i H_j + \frac{1}{2} \delta_{ij} (\vec{\mathbf{B}} \cdot \vec{\mathbf{H}} + \vec{\mathbf{D}} \cdot \vec{\mathbf{E}})$$

$$f_j = \partial_k [D_k E_j + B_k H_j - \frac{1}{2} \delta_{jk} (\vec{\mathbf{B}} \cdot \vec{\mathbf{H}} + \vec{\mathbf{D}} \cdot \vec{\mathbf{E}})] = -\partial_k T_{kj}$$

Resonant Cavities

$$\vec{\mathbf{H}}_{0,z} = \frac{1}{\epsilon\mu\omega^2 - k^2} \left( \vec{\nabla}_t \frac{\partial H_{0,z}}{\partial z} + i\omega\epsilon \hat{\mathbf{z}} \times \vec{\nabla}_t E_{0,z} \right)$$

$$\vec{\mathbf{E}}_{0,z} = \frac{1}{\epsilon\mu\omega^2 - k^2} \left( \vec{\nabla}_t \frac{\partial E_{0,z}}{\partial z} - i\mu\omega \hat{\mathbf{z}} \times \vec{\nabla}_t H_{0,z} \right)$$

**Some useful mathematical information:**

1. General solution of  $\frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta) \frac{\partial V}{\partial \theta} = 0$  in spherical coordinates

$$V(r, \theta) = \sum_{n=0}^{\infty} (A_n r^n + \frac{B_n}{r^{n+1}}) P_n(\cos\theta)$$

where  $P_n(\cos\theta)$  is the Legendre polynomials and

$$\int_{-\pi}^{\pi} P_m(\cos\theta) P_n(\cos\theta) d(\cos\theta) = \frac{2}{2n+1} \delta_{mn}$$

$$P_0(\cos\theta) = 1, P_1(\cos\theta) = \cos\theta, P_2(\cos\theta) = (3\cos^2\theta - 1)/2, P_3(\cos\theta) = (5\cos^3\theta - 3\cos\theta)/2,$$

2. General solution of  $\frac{\partial}{\partial \rho}(\rho \frac{\partial V}{\partial \rho}) + \frac{1}{\rho} \frac{\partial^2 V}{\partial \phi^2} = 0$  in cylindrical coordinates

$$V(\rho, \phi) = D_0 + A_0 \ln \rho + \sum_v (A_v \rho^v + \frac{B_v}{\rho^v}) \{C_v \sin v\phi + D_v \cos v\phi\}$$

For unrestricted  $\phi$ ,  $v =$  positive non-zero integer.

$$\int_{-\pi}^{\pi} \sin m\phi \sin n\phi d\phi = \pi \delta_{mn}, \int_{-\pi}^{\pi} \cos m\phi \cos n\phi d\phi = \pi \delta_{mn}, \int_{-\pi}^{\pi} \sin m\phi \cos n\phi d\phi = 0$$

3. General solution of  $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ ,  $V(x, y) = \sum a_n \{ \exp(\pm i a_n x) \} \{ \exp(\pm i a_n y) \}$

$$e^{\pm i x} = \cos x \pm i \sin x, e^{\pm x} = \cosh x \pm \sinh x$$

$$\int_0^a \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{a} dx = \frac{a}{2} \delta_{mn}, \int_0^a \cos \frac{m\pi x}{a} \cos \frac{n\pi x}{a} dx = \frac{a}{2} \delta_{mn}, \int_0^a \sin \frac{m\pi x}{a} \cos \frac{n\pi x}{a} dx = 0,$$

4. Binomial Theorem :  $(1 + x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \dots$

5. Some frequently used integrals :

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 + x^2}}, \int \frac{dx}{(a^2 + x^2)^{1/2}} = \ln(x + \sqrt{a^2 + x^2})$$

$$\int [\sqrt{a^2 + x^2}]^{\pm n} x dx = \frac{1}{2 \pm n} [\sqrt{a^2 + x^2}]^{2 \pm n} \quad (n \neq 2 \text{ or } -2)$$

$$\int \frac{x^2 dx}{(a^2 + x^2)^{3/2}} = \frac{-x}{\sqrt{a^2 + x^2}} + \ln(x + \sqrt{a^2 + x^2})$$

6. Levi-Civita Tensors :  $\epsilon_{ijk} \epsilon_{lmn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$ ,

$$\epsilon_{ijk} \epsilon_{ijn} = 2 \delta_{kn}$$

$$(A \times B)_i = \epsilon_{ijk} A_j B_k$$

## VECTOR DERIVATIVES

CARTESIAN.  $dl = dx \hat{i} - dy \hat{j} - dz \hat{k}$ ;  $d\tau = dx dy dz$

Gradient.  $\nabla t = \frac{\partial t}{\partial x} \hat{i} - \frac{\partial t}{\partial y} \hat{j} - \frac{\partial t}{\partial z} \hat{k}$

Divergence.  $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl.  $\nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{i} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{j} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{k}$

Laplacian.  $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

SPHERICAL.  $dl = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$ ;  $d\tau = r^2 \sin \theta dr d\theta d\phi$

Gradient.  $\nabla t = \frac{\partial t}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\phi}$

Divergence.  $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$

Curl.  $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r}$   
 $+ \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi}$

Laplacian.  $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$

CYLINDRICAL.  $dl = dr \hat{r} + r d\phi \hat{\phi} + dz \hat{z}$ ;  $d\tau = r dr d\phi dz$

Gradient.  $\nabla t = \frac{\partial t}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial t}{\partial \phi} \hat{\phi} + \frac{\partial t}{\partial z} \hat{z}$

Divergence.  $\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl.  $\nabla \times \mathbf{v} = \left[ \frac{1}{r} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{r} + \left[ \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right] \hat{\phi}$   
 $+ \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{z}$

Laplacian.  $\nabla^2 t = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial t}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

## FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12} \text{ coul}^2/\text{N}\cdot\text{m}^2$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$	(permeability of free space)
$c = 3.00 \times 10^8 \text{ m/sec}$	(speed of light)
$e = 1.60 \times 10^{-19} \text{ coul}$	(charge of the electron)
$m = 9.11 \times 10^{-31} \text{ kg}$	(mass of the electron)

## CONVERSION FROM SPHERICAL TO CARTESIAN COORDINATES

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad \begin{cases} \hat{i} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \\ \hat{j} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \\ \hat{k} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \end{cases}$$
  

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \quad \begin{cases} \hat{r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \\ \hat{\theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \\ \hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j} \end{cases}$$