Preliminary Exam on Electricity and Magnetism (spring 2000)
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Do all problems. The time limit is 3 hours. Please start each problem on a new sheet and show all work. If you use more than one blue book, sign all books and indicate the problem numbers on the front page.

There are 1 sheet of exam problems, 2 sheets of formulae and 4 sheets of math formulae. SI units are preferred but not required. If you cannot solve a problem completely, demonstrate that you are on the right track.

1. (10 pts) Consider a long solenoid of radius a with n turns/unit length of wires wrapped around its circumference. Suppose the current (I) in the wires is increasing linearly with time, (i.e., $dI/dt = K =$ constant). Find the electric field **E** inside and outside the solenoid.

2. (15 pts) Two concentric spherical shells have inner and outer radii of a and b respectively. At $r = a$, the potential is $V = V_0 \cos\theta$ where V_0 is a constant, and θ is the angle with respect to an axis through the center. At $r = b$, the potential $V = 0$. Find the potential in the space between the shells ($a \le r \le b$).

3. (15 pts) Two point charges q and $-3q$ are separated by a distance S. Find the equipotential surface with zero potential.

4. (20 pts) We wish to find out the fraction of light intensity transmitted at normal incidence through glass for those who wear glasses. We assume that the light is linearly polarized, monochromatic with a wavelength λ and intensity W. Light transmission through air (index of refraction $n_1 = 1$) and glass (index of refraction $n_2 = 1.3$) are assumed to be without loss. The magnetic permeability of air and glass can be taken to be 1.
(a) What should be appropriate boundary conditions at the air/glass interface ?
(b) Find the expressions for the **E** part and **B** (or **H**) part of the electromagnetic wave in air and those in glass.
(c) What is the fraction of light that transmits through the glasses and enters the eyes?

5. (20 pts) An infinitely long thin wire carrying a DC current I is surrounded coaxially by a cylindrical shell of inner radius a and outer radius b, of a linear magnetic material of constant susceptibility χ.
(a) Find **B**, **H** and **M** everywhere
(b) Find the bound current density (or magnetization current density), both volume current density ($\nabla \times \mathbf{M}$) and surface current ($\mathbf{M} \times \mathbf{n}_1$).

6. (20 pts) A conducting sphere of radius R carries a total charge of Q, which resides on the surface. The sphere is spinning about a diameter with an angular velocity ω.
(a) Find the magnetic induction (**B** field) at the center of the sphere.
(b) Find the effective magnetic moment and its direction for the spinning charged sphere.
At large distances from the sphere ($r \gg R$), what will be a reasonable form for the
(c) magnetic induction (**B**)
(d) the vector potential (**A**), where $\mathbf{B} = \nabla \times \mathbf{A}$?

Some Physics Formulae:

Electric dipole

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}$$

$$\mathbf{E}_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}[2\cos\theta\, \hat{\mathbf{r}}_1 + \sin\theta\, \hat{\boldsymbol{\theta}}_1]$$

For a dielectric with polarization **P**, the potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{-\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' + \frac{1}{4\pi\varepsilon_0}\int\frac{\mathbf{P}(\mathbf{r}')\cdot\mathbf{n}_1}{|\mathbf{r}-\mathbf{r}'|}da'$$

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

Quadrupole moment tensor : $Q_{ij} = \int (3\, x_i x_j - r^2\,\delta_{ij})\,\rho(\mathbf{r})\, d\tau$

Current density : $\mathbf{J} = n\, q\, \mathbf{v} = \rho\, \mathbf{v}$

Ohm's law : $\mathbf{J} = \sigma\,\mathbf{E}$

Capacitor : $Q = CV$

Continuity Equation : $\nabla\cdot\mathbf{J} + \frac{\partial\rho_f}{\partial t} = 0,$

Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{I\, d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_f(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau'$$

Vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_f(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau'$$

Axial field due to a current loop of radius a and current I

$$B = \frac{\mu_0\, I\,(\pi a^2)}{2\pi\,(a^2+z^2)^{3/2}}$$

Magnetic dipole

$$\mathbf{A}_{\text{dipole}} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}$$

$$\mathbf{B}_{\text{dipole}} = \frac{\mu_0}{4\pi} \frac{m}{r^3} [2\cos\theta\, \hat{\mathbf{r}}_1 + \sin\theta\, \hat{\boldsymbol{\theta}}_1]$$

For a magnetized object

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \mathbf{n}_1}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$$

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{-\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{1}{4\pi} \int \frac{\mathbf{n}_1 \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

Lorentz condition : $\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0$

Plane light wave propagating in $\mathbf{n}_1$ (unit vector) direction

$$\mathbf{B} = \frac{1}{v} \mathbf{n}_1 \times \mathbf{E} = \frac{k}{\omega} (\mathbf{n}_1 \times \mathbf{E}), \qquad v^2 = \frac{1}{\mu\varepsilon},$$

index of refraction $\quad n = \frac{c}{v}$

Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Time-average power density $\; P = |\langle S \rangle| = \frac{1}{2} \operatorname{Re} |\mathbf{E} \times \mathbf{H}^*| = \frac{1}{2} v\, \varepsilon\, |E_0|^2$

Maxwell stress tensor: $T_{ij} = -D_i E_j - B_i H_j + \frac{1}{2} \delta_{ij} (\vec{B} \cdot \vec{H} + \vec{D} \cdot \vec{E}).$

$$f_j = \partial_k \left[D_k E_j + B_k H_j - \tfrac{1}{2} \delta_{jk} (\vec{B} \cdot \vec{H} + \vec{D} \cdot \vec{E}) \right] = -\partial_k T_{kj}$$

Resonant Cavities

$$\vec{H}_{0,t} = \frac{1}{\varepsilon\mu\omega^2 - k^2} \left(\vec{\nabla}_t \frac{\partial H_{0,z}}{\partial z} + i\omega\varepsilon \hat{k} \times \vec{\nabla}_t E_{0,z} \right)$$

$$\vec{E}_{0,t} = \frac{1}{\varepsilon\mu\omega^2 - k^2} \left(\vec{\nabla}_t \frac{\partial E_{0,z}}{\partial z} - i\mu\omega \hat{k} \times \vec{\nabla}_t H_{0,z} \right)$$

Some useful mathematical information:

1. General solution of $\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial V}{\partial\theta} = 0$ in spherical coordinates

$$V(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + \frac{B_n}{r^{n+1}})\, P_n(\cos\theta)$$

where $P_n(\cos\theta)$ is the Legendre polynomials and

$$\int_{-\pi}^{\pi} P_m(\cos\theta)\, P_n(\cos\theta)\, d(\cos\theta) = \frac{2}{2n+1}\,\delta_{mn}$$

$P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = (3\cos^2\theta-1)/2$, $P_3(\cos\theta) = (5\cos^3\theta-3\cos\theta)/2$,

2. General solution of $\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho}) + \frac{1}{\rho}\frac{\partial^2 V}{\partial\phi^2} = 0$ in cylindrical coordinates

$$V(\rho,\phi) = D_o + A_o\, ln\rho + \sum_{\nu}(A_\nu\rho^\nu + \frac{B_\nu}{\rho^\nu})\,\{C_\nu \sin\nu\phi + D_\nu\cos\nu\phi\}$$

For unrestricted ϕ, ν = positive non-zero integer.

$$\int_{-\pi}^{\pi}\sin m\phi\,\sin n\phi\, d\phi = \pi\,\delta_{mn},\quad \int_{-\pi}^{\pi}\cos m\phi\,\cos n\phi\, d\phi = \pi\,\delta_{mn},\quad \int_{-\pi}^{\pi}\sin m\phi\,\cos n\phi\, d\phi = 0$$

3. General solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, $\quad V(x,y) = \sum_{a_n}\{\exp(\pm a_n x)\}\,\{\exp(\pm i a_n y)\}$

$e^{\pm ix} = \cos x \pm i\sin x$, $\quad e^{\pm x} = \cosh x \pm \sinh x$

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn},\quad \int_0^a \cos\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn},\quad \int_0^a \sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = 0,$$

4. Binomial Theorem : $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$

5. Some frequently used integrals :

$$\int\frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}},\quad \int\frac{dx}{(a^2+x^2)^{1/2}} = ln(x+\sqrt{a^2+x^2})$$

$$\int[\sqrt{a^2+x^2}\,]^{\pm n}\, x\,dx = \frac{1}{2\pm n}[\sqrt{a^2+x^2}\,]^{2\pm n}\quad (n\neq 2 \text{ or } -2)$$

$$\int\frac{x^2dx}{(a^2+x^2)^{3/2}} = \frac{-x}{\sqrt{a^2+x^2}} + ln(x+\sqrt{a^2+x^2})$$

6. Levi-Civita Tensors : $\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$, $\qquad \varepsilon_{ijk}\,\varepsilon_{ijn} = 2\,\delta_{kn}$

$(A \times B)_i = \varepsilon_{ijk}\, A_j\, B_k$,

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$; $d\tau = dx\,dy\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial x}\hat{i} + \frac{\partial t}{\partial y}\hat{j} + \frac{\partial t}{\partial z}\hat{k}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

Laplacian. $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl. $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

Laplacian. $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

CYLINDRICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$; $d\tau = r\,dr\,d\phi\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial\phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial\phi}\right]\hat{z}$$

Laplacian. $$\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}$ coul^2/N-m^2	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}$ N/A^2	(permeability of free space)
$c = 3.00 \times 10^{8}$ m/sec	(speed of light)
$e = 1.60 \times 10^{-19}$ coul	(charge of the electron)
$m = 9.11 \times 10^{-31}$ kg	(mass of the electron)

CONVERSION FROM SPHERICAL TO CARTESIAN COORDINATES

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{i} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{j} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ \hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} \\ \hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \end{cases}$$