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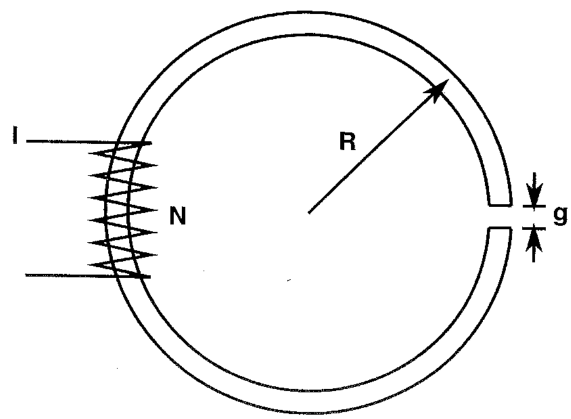
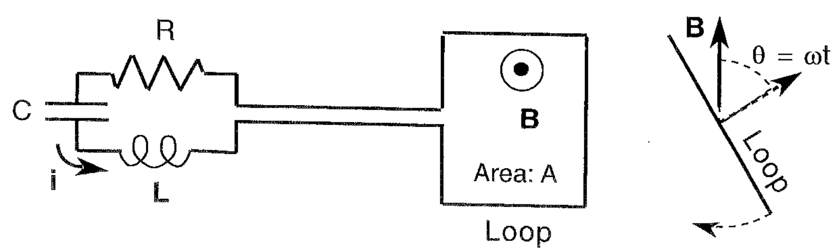
Preliminary Exam: Electricity and Magnetism

This is a closed book exam. Page 5 is a sheet of useful equations. Everything that you need is derivable from them in at most a few steps. Please attempt all five problems. Good luck!

1. Consider an iron toroid of radius R and permeability $\mu(H)$. A current I energizes an N-turn coil wrapped about the toroid. Calculate the magnetic (B) field in the gap of size g. You may ignore fringing effects at the edges of the gap.



2. A single turn coil of area A is placed in a uniform magnetic field $\vec{B} = B_0\hat{y}$. The coil is rotated in the field at angular velocity ω so that the angle between the field and the normal to the coil is $\theta = \omega t$ at time t. The coil is connected to an L-R-C circuit as shown below.



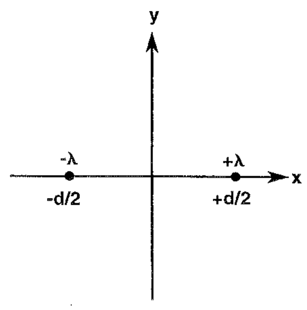
(a) Write down the differential equation describing the voltage V_c across the capacitor as a function of time t. (Be very careful with signs other than overall ones.)

(b) Solve the differential equation.

(c) What is the form of the solution in the low frequency limit?

(d) Find the finite frequency at which the phase of the solution shifts by $\pi/2$ with respect to the low frequency solution.

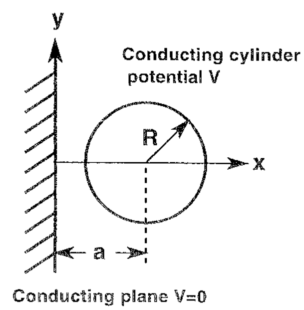
3. Consider the case of two infinitely long charged wires of uniform linear charge density $+\lambda$ and $-\lambda$ separated by a distance d.



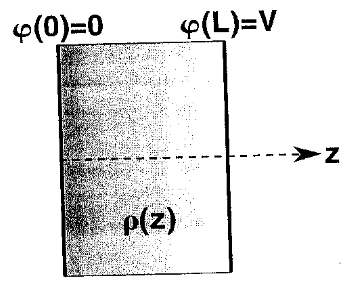
(a) Show that the equipotential surfaces are either a plane for potential $\varphi(x, y) = 0$ or a cylinder for $\varphi(x, y) \neq 0$.

(b) Calculate coordinates of the cylinder axis and the radius of the cylinder for $\varphi(x, y) = V$.

(c) Use these results to calculate the surface charge density on a conducting plane that is located a distance a from a conducting cylinder of radius R that is held at potential V with respect to the plane.



4. A parallel plate capacitor consists of an cathode at $z = 0$ held at potential $\varphi(0) = 0$ and an anode at $z = L$ held at potential $\varphi(L) = V$ (ignore edge effects). The cathode emits a steady stream of electrons so that the entire gap is filled with a uniform current density J (J is a constant independent of z). The field from the electron cloud cancels the electric field at the cathode so that electrons are emitted with zero velocity.



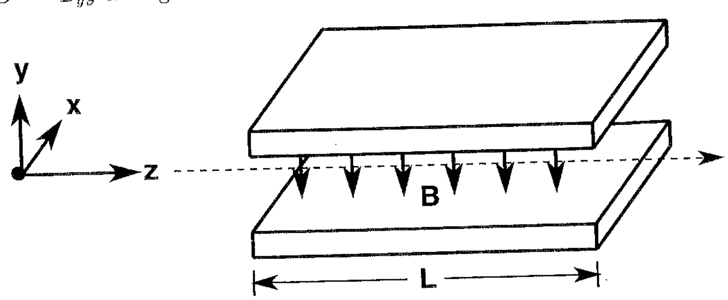
(a) Calculate the electron velocity at some point in the gap in terms of the potential $\varphi(z)$ and use this result to express the charge density $\rho(z)$ in terms of J and $\varphi(z)$.

(b) Using the charge density from part (a), calculate the potential in the gap. Express $\varphi(z)$ as a function of z, V, and L. Hint: note that

$$\frac{d^2\varphi}{dz^2} = \frac{1}{2}\frac{d}{d\varphi}\left(\frac{d\varphi}{dz}\right)^2$$

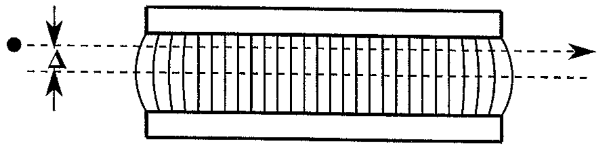
(c) Express J in terms of V and L.

5. A particle of charge q and momentum $\vec{p} = p_z\hat{z}$ is incident on a uniform vertical magnetic field $\vec{B} = -B_y\hat{y}$ of length L.



(a) Using the small angle approximation $p_z \simeq |\vec{p}|$ for the entire passage through the field, estimate the horizontal momentum p_x and bend angle θ_x caused by the magnetic field.

(b) Maxwell's equations require that all dipole magnets have some external fringe fields. If the particle trajectory begins displaced above the midplane of the magnet by a distance Δ, it will encounter small B_z components as it passes through the fringe fields.



The small p_x component generated by the magnet can interact with the z-component of the fringe field as the particle exits the magnet. This generates a very small vertical momentum component p_y. Express p_y in terms of the integral $\int dz B_z$ where the lower limit of the integration is inside the magnet (where the field is vertical) and the upper limit is at $z = \infty$.

(c) Evaluate the integral in terms of the vertical field B_y in the magnet and the displacement Δ. Express p_y and θ_y in terms of B_y, L, Δ, and p_z.

(d) Does the fringe field focus charged particles toward the midplane of the magnet or defocus them away from the midplane?

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}$ coul²/N-m²	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}$ N/A²	(permeability of free space)
$c = 3.00 \times 10^{8}$ m/sec	(speed of light)
$e = 1.60 \times 10^{-19}$ coul	(charge of the electron)
$m = 9.11 \times 10^{-31}$ kg	(mass of the electron)

CONVERSION FROM SPHERICAL TO CARTESIAN COORDINATES

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{i} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{j} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ \hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} \\ \hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \end{cases}$$

Some useful mathematical information:

1. General solution of $\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial V}{\partial\theta} = 0$ in spherical coordinates

$$V(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + \frac{B_n}{r^{n+1}}) P_n(\cos\theta)$$

where $P_n(\cos\theta)$ is the Legendre polynomials and

$$\int_{-\pi}^{\pi} P_m(\cos\theta)\, P_n(\cos\theta)\, d(\cos\theta) = \frac{2}{2n+1}\delta_{mn}$$

$P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$, $P_3(\cos\theta) = (5\cos^3\theta - 3\cos\theta)/2$,

2. General solution of $\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho}) + \frac{1}{\rho}\frac{\partial^2 V}{\partial\phi^2} = 0$ in cylindrical coordinates

$$V(\rho,\phi) = D_o + A_o\, ln\rho + \sum_{\nu}(A_\nu \rho^\nu + \frac{B_\nu}{\rho^\nu})\{C_\nu \sin\nu\phi + D_\nu \cos\nu\phi\}$$

For unrestricted ϕ, ν = positive non-zero integer.

$$\int_{-\pi}^{\pi}\sin m\phi\, \sin n\phi\, d\phi = \pi\,\delta_{mn},\quad \int_{-\pi}^{\pi}\cos m\phi\, \cos n\phi\, d\phi = \pi\,\delta_{mn},\quad \int_{-\pi}^{\pi}\sin m\phi\, \cos n\phi\, d\phi = 0$$

3. General solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, $\quad V(x,y) = \sum_{a_n}\{\exp(\pm a_n x)\}\{\exp(\pm i a_n y)\}$

$e^{\pm ix} = \cos x \pm i\sin x$, $\quad e^{\pm x} = \cosh x \pm \sinh x$

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn},\quad \int_0^a \cos\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn},\quad \int_0^a \sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = 0,$$

4. Binomial Theorem : $(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + ...$

5. Some frequently used integrals :

$$\int\frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}},\quad \int\frac{dx}{(a^2+x^2)^{1/2}} = ln(x + \sqrt{a^2+x^2})$$

$$\int[\sqrt{a^2+x^2}\,]^{\pm n}\, x\,dx = \frac{1}{2\pm n}[\sqrt{a^2+x^2}\,]^{2\pm n}\quad (n \neq 2 \text{ or } -2)$$

$$\int\frac{x^2 dx}{(a^2+x^2)^{3/2}} = \frac{-x}{\sqrt{a^2+x^2}} + ln(x + \sqrt{a^2+x^2})$$

6. Levi-Civita Tensors : $\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$, $\qquad \varepsilon_{ijk}\,\varepsilon_{ijn} = 2\,\delta_{kn}$

$(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk}\, A_j\, B_k$,