Electricity and Magnetism
Do all 7 problems

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Preliminary Exam (~~October~~ 29, 1996)

1. (10 points) Derive and sketch the charge distribution which will give rise to a Yukawa potential

$$\psi = \frac{e^{-ra}}{r}$$

2. (10 points) A long wire of radius a is supported parallel to and at a height h above the earth. Assume that the surface of the earth is a conducting plane. If the potential difference between the wire and the earth is V,

a) Find the charge per unit length on the wire.

b) Find the electric field intensity directly under the wire at the surface of the earth.

3. (15 points) A circular disk rotates about its axis with angular velocity ω. The disk is made of metal with conductivity g, and its thickness is t. The rotating disk is placed between the pole faces of a magnet which produces a uniform magnetic field **B** over a small square area of size a^2 at the average distance r from the axis; **B** is perpendicular to the disk. Find the approximate torque on the disk.

4. (15 points) A resistive cylinder of radius a with a narrow longitudinal gap (see the figure below) at $\phi = \pi$ carries a current in the azimuthal direction, giving rise to a linearly varying potential $V(a,\phi) = V_0\phi/2\pi$ for $-\pi < \phi < \pi$. Find the potential inside the cylinder.

5. (15 points) A pair of concentric cylinders of radii a and b and of the same length are connected to the terminals of a battery supplying an EMF V (see the figure below). Find the force in the axial direction on the inner cylinder when it is partially extracted from the outer one.

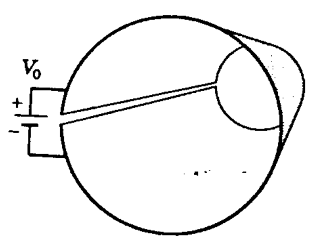

Figure for Prob. 4

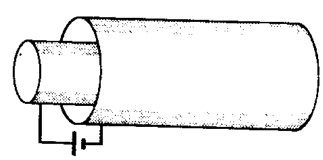

Figure for Prob. 5

6. (15 points) Consider a permanent magnet in the shape of a solid sphere of radius R. The magnetization is constant throughout the interior of the magnet. Find **B** and **H** everywhere. Sketch the line of force for **B** and **H** everywhere denoting the salient features.

7. (20 points) A rectangular cavity of dimension a, a, L, and walls of perfect conductivity is excited in the mode.

$$E_z = E_0 \sin(\pi x/a) \sin(\pi y/a) e^{-i\omega t}$$

$$H_z = E_x = E_y = 0$$

What is ω? Calculate the forces exerted on all the walls.

Some Physics Formulae:

Electric dipole

$$V_{dipole} = \frac{1}{4\pi\varepsilon_o} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$$

$$E_{dipole} = \frac{1}{4\pi\varepsilon_o} \frac{p}{r^3} [2\cos\theta\, \hat{\mathbf{r}}_1 + \sin\theta\, \hat{\boldsymbol{\theta}}_1]$$

For a dielectric with polarization **P**, the potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'$$

$$= \frac{1}{4\pi\varepsilon_o} \int \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' + \frac{1}{4\pi\varepsilon_o} \int \frac{\mathbf{P}(\mathbf{r}') \cdot \mathbf{n}_1}{|\mathbf{r} - \mathbf{r}'|} da'$$

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P}$$

Quadrupole moment tensor : $Q_{ij} = \int (3\, x_i x_j - r^2 \delta_{ij})\, \rho(\mathbf{r})\, d\tau$

Current density : $\mathbf{J} = n\, q\, \mathbf{v} = \rho\, \mathbf{v}$

Ohm's law : $\mathbf{J} = \sigma\, \mathbf{E}$

Capacitor : $Q = CV$

Continuity Equation : $\nabla \cdot \mathbf{J} + \frac{\partial \rho_f}{\partial t} = 0,$

Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int \frac{I\, d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_o}{4\pi} \int \frac{\mathbf{J}_f(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'$$

Vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int \frac{\mathbf{J}_f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau'$$

Axial field due to a current loop of radius a and current I

$$B = \frac{\mu_o I (\pi a^2)}{2\pi (a^2 + z^2)^{3/2}}$$

Magnetic dipole

$$\mathbf{A}_{\text{dipole}} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}$$

$$\mathbf{B}_{\text{dipole}} = \frac{\mu_0}{4\pi} \frac{m}{r^3} [2\cos\theta\, \hat{\mathbf{r}}_1 + \sin\theta\, \hat{\theta}_1]$$

For a magnetized object

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \mathbf{n}_1}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$$

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{-\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d\tau' + \frac{1}{4\pi} \int \frac{\mathbf{n}_1 \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} da'$$

Lorentz condition : $\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial\phi}{\partial t} = 0$

Plane light wave propagating in $\mathbf{n}_1$ (unit vector) direction

$$\mathbf{B} = \frac{1}{v} \mathbf{n}_1 \times \mathbf{E} = \frac{k}{\omega} (\mathbf{n}_1 \times \mathbf{E}), \qquad v^2 = \frac{1}{\mu\varepsilon},$$

index of refraction $n = \frac{c}{v}$

Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Time-average power density $\mathbf{P} = |\langle S \rangle| = \frac{1}{2} \text{Re}\, |\mathbf{E} \times \mathbf{H}^*| = \frac{1}{2} v\, \varepsilon\, |E_o|^2$

Maxwell stress tensor: $T_{ij} = -D_i E_j - B_i H_j + \frac{1}{2}\delta_{ij}(\vec{B} \cdot \vec{H} + \vec{D} \cdot \vec{E}).$

$$f_j = \partial_k \left[D_k E_j + B_k H_j - \tfrac{1}{2}\delta_{jk}\left(\vec{B} \cdot \vec{H} + \vec{D} \cdot \vec{E}\right)\right] = -\partial_k \vec{T}_{kj}$$

Resonant Cavities

$$\vec{H}_{0,t} = \frac{1}{\varepsilon\mu\omega^2 - k^2}\left(\vec{\nabla}_t \frac{\partial H_{0,z}}{\partial z} + i\omega\varepsilon \hat{k} \times \vec{\nabla}_t E_{0,z}\right)$$

$$\vec{E}_{0,t} = \frac{1}{\varepsilon\mu\omega^2 - k^2}\left(\vec{\nabla}_t \frac{\partial E_{0,z}}{\partial z} - i\mu\omega \hat{k} \times \vec{\nabla}_t H_{0,z}\right)$$

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b (\nabla f)\cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_{\text{volume}} (\nabla\cdot\mathbf{A})\, d\tau = \oint_{\text{surface}} \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int_{\text{surface}} (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint_{\text{line}} \mathbf{A}\cdot d\mathbf{l}$

Some useful mathematical information:

1. General solution of $\frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta) \frac{\partial V}{\partial\theta} = 0$ in spherical coordinates

$$V(r,\theta) = \sum_{n=0}^{\infty} (A_n r^n + \frac{B_n}{r^{n+1}}) P_n(\cos\theta)$$

where $P_n(\cos\theta)$ is the Legendre polynomials and

$$\int_{-\pi}^{\pi} P_m(\cos\theta) P_n(\cos\theta) d(\cos\theta) = \frac{2}{2n+1} \delta_{mn}$$

$P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = (3\cos^2\theta-1)/2$, $P_3(\cos\theta) = (5\cos^3\theta-3\cos\theta)/2$,

2. General solution of $\frac{\partial}{\partial\rho}(\rho \frac{\partial V}{\partial\rho}) + \frac{1}{\rho} \frac{\partial^2 V}{\partial\phi^2} = 0$ in cylindrical coordinates

$$V(\rho,\phi) = D_o + A_o\, ln\rho + \sum_{\nu} (A_\nu \rho^\nu + \frac{B_\nu}{\rho^\nu}) \{C_\nu \sin\nu\phi + D_\nu \cos\nu\phi\}$$

For unrestricted ϕ, ν = positive non-zero integer.

$$\int_{-\pi}^{\pi} \sin m\phi \sin n\phi\, d\phi = \pi\, \delta_{mn}, \quad \int_{-\pi}^{\pi} \cos m\phi \cos n\phi\, d\phi = \pi\, \delta_{mn}, \quad \int_{-\pi}^{\pi} \sin m\phi \cos n\phi\, d\phi = 0$$

3. General solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, $\quad V(x,y) = \sum_{a_n} \{\exp(\pm a_n x)\} \{\exp(\pm i a_n y)\}$

$e^{\pm ix} = \cos x \pm i \sin x$, $\quad e^{\pm x} = \cosh x \pm \sinh x$

$$\int_0^a \sin\frac{m\pi x}{a} \sin\frac{n\pi x}{a} dx = \frac{a}{2} \delta_{mn}, \quad \int_0^a \cos\frac{m\pi x}{a} \cos\frac{n\pi x}{a} dx = \frac{a}{2} \delta_{mn}, \quad \int_0^a \sin\frac{m\pi x}{a} \cos\frac{n\pi x}{a} dx = 0,$$

4. Binomial Theorem : $(1 + x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \ldots$

5. Some frequently used integrals :

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}}, \quad \int \frac{dx}{(a^2 + x^2)^{1/2}} = ln(x + \sqrt{a^2 + x^2})$$

$$\int [\sqrt{a^2 + x^2}\,]^{\pm n} x dx = \frac{1}{2 \pm n} [\sqrt{a^2 + x^2}\,]^{2 \pm n} \quad (n \neq 2 \text{ or } -2)$$

$$\int \frac{x^2 dx}{(a^2 + x^2)^{3/2}} = \frac{-x}{\sqrt{a^2 + x^2}} + ln(x + \sqrt{a^2 + x^2})$$

6. Levi-Civita Tensors : $\varepsilon_{ijk}\, \varepsilon_{imn} = \delta_{jm}\, \delta_{kn} - \delta_{jn}\, \delta_{km}$, $\quad \varepsilon_{ijk}\, \varepsilon_{ijn} = 2\, \delta_{kn}$

$(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk} A_j B_k$,

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$; $d\tau = dx\,dy\,dz$

Gradient. $\nabla t = \frac{\partial t}{\partial x}\hat{\imath} + \frac{\partial t}{\partial y}\hat{\jmath} + \frac{\partial t}{\partial z}\hat{k}$

Divergence. $\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl. $\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$

Laplacian. $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\phi}$

Divergence. $\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Curl. $\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r}$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

Laplacian. $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

CYLINDRICAL $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$; $d\tau = r\,dr\,d\phi\,dz$

Gradient. $\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial\phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$

Divergence. $\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl. $\nabla\times\mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi}$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial\phi}\right]\hat{z}$$

Laplacian. $\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$