September, 1993 Preliminary Exam on Electricity and Magnetism

This is a 3 hour exam with no additional literature or notes allowed
Please make your line of reasoning clear in your solution. Do all problems starting each on a fresh page. Good luck!

Problem 1 (~25%)

(a) Suppose the electrical potential (V) depends only on plane polar coordinates (r, ϕ), where $r^2 = x^2 + y^2$ and $\tan \phi = y/x$, x and y are the Cartersian coordinates. Show that the solution to the Laplace equation ($\nabla^2 V = 0$) has the general form of

$$V(r, \phi) = a_o + b_o \ln r + \sum_{n=1}^{\infty} [a_n r^n \sin(n\phi + \alpha_n) + b_n r^{-n} \sin(n\phi + \beta_n)]$$

where a_m, b_m, α_m, β_m are constants.

(b) Now consider an infinitely long hollow conducting circular cylinder, which is split length wise into two equal halves held at potentials $+ V_o$ and $- V_o$. The radius of the cylinder is r_o. Find the potential inside and outside the cylinder.

(c) Calculate the charge density on the cylinder as a function of ϕ. If your answer is in the form of an infinite series, reduce the series to a simple term.

Problem 2 (~15%)

A dielectric sphere of radius R_o has a permanent polarization of $\mathbf{P} = f(r)\mathbf{r}$, where $f(r)$ is a scalar function of r and $\mathbf{r}$ is a vector measured from the center of the sphere.

(a) Find the volume bound charge density and surface bound charge density.
(b) What is the total amount of bound charge?
(c) Find the electric field (**E**) and the displacement field (**D**) everywhere.

Problem 3 (~20%)

A conductor has the shape of an infinite plane with a hemispherical bulge of radius a as shown. A positive point charge +Q is placed above the center of the bulge at distance d from the plane ($d > a$). The conducting plane (and the bulge) is maintained at zero potential.

(a) What is the force on the positive point charge +Q?
(b) Is the point charge attracted to, or repelled from, the infinite sheet?
(c) Answer part (b) if the point charge is negative (-Q).

Problem 4 (~15%)

Consider two co-axial, parallel conducting rings, a and b, of radii R_a and R_b respectively. The centers of the rings are separated by a distance L which far exceeds R_a ($L >> R_a$) but <u>not</u> R_b ($L \approx R_b$) as shown in the figure.

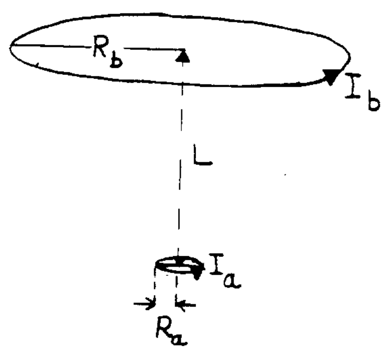

Figure 1

(a) Show that the mutual inductance of the rings is

$$M = \frac{\mu_0 \pi}{2} \frac{R_a^2 R_b^2}{(L^2 + R_b^2)^{3/2}}$$

(b) Currents I_a and I_b pass through rings a and b respectively in the same sense. Calculate the direction and magnitude of the force between the rings.

Problem 5 (~25%)

In plane wave solutions to Maxwells equations the electric and magnetic fields $\tilde{\mathbf{E}}$ & $\tilde{\mathbf{H}}$ can be written

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \mathrm{Re}\left\{\mathbf{E}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right\}$$

$$\tilde{\mathbf{H}}(\mathbf{r}, t) = \mathrm{Re}\left\{\mathbf{H}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right\}$$

where E, H and k in general are complex vectors.

(a) **Consider a linear homogeneous non-conducting medium with permeability μ and permittivity ε. Write down Maxwells equations as vector equations involving E, H, k, μ, ε and ω.**

(b) **Using these equations show that**

$$(\mathbf{k}')^2 - (\mathbf{k}'')^2 = \mu\varepsilon\omega^2$$

$$2\mathbf{k}' \cdot \mathbf{k}'' = 0$$

where $\mathbf{k} = \mathbf{k}' + i\mathbf{k}''$ and $\mathbf{k}'$, $\mathbf{k}''$ are real vectors.

Consider the total internal reflection of a plane wave at the boundary between two dielectrics as shown in the figure below.

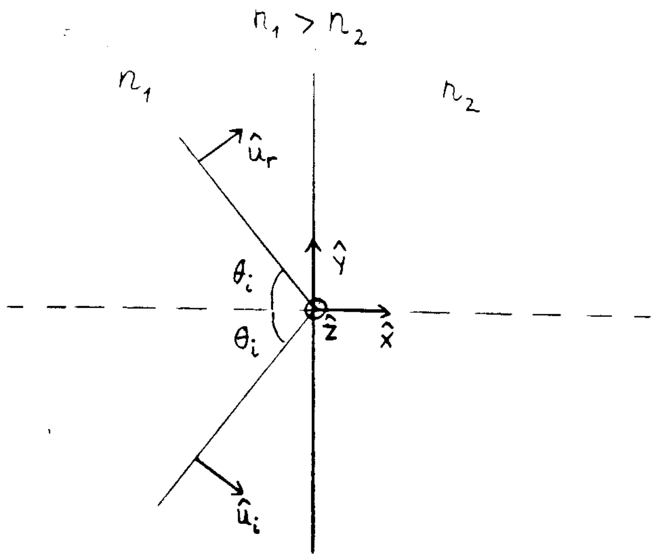

Figure 2

Both dielectrics have the vacuum permeability but different indexes of refraction $n_1 > n_2$. The angle of incidence exceeds the critical angle for total internal reflection: $\sin\theta_i > \sin\theta_c = n_2/n_1$

(c) **Determine the complex wavevector $\mathbf{k}_2 = \mathbf{k}_2' + i\,\mathbf{k}_2''$ associated with the evanescent wave in medium 2.**

(d) **All electric fields in this problem are polarized normal to the plane of incidence. E_i is the amplitude of the electric field in the incident plane wave. Show that the amplitude of the electric field in the evanescent wave at the boundary, E_t, is given by**

$$\left(\frac{E_t}{E_i}\right)_\perp = \frac{2\cos\theta_i}{\cos\theta_i + i\sqrt{\sin^2\theta_i - \sin^2\theta_c}}$$

VECTOR FORMULAS

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \quad (1\text{-}29)$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \quad (1\text{-}30)$$

$$\nabla\times\nabla u = 0 \quad (1\text{-}48)$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \quad (1\text{-}49)$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \quad (1\text{-}106)$$

$$\frac{d}{d\sigma}(u\mathbf{A}) = \frac{du}{d\sigma}\mathbf{A} + u\frac{d\mathbf{A}}{d\sigma} \quad (1\text{-}107)$$

$$\frac{d}{d\sigma}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{d\sigma}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{d\sigma} \quad (1\text{-}108)$$

$$\frac{d}{d\sigma}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{d\sigma}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{d\sigma} \quad (1\text{-}109)$$

$$\nabla(u+v) = \nabla u + \nabla v \quad (1\text{-}110)$$

$$\nabla(uv) = u\nabla v + v\nabla u \quad (1\text{-}111)$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} \quad (1\text{-}112)$$

$$\nabla(\mathbf{C}\cdot\mathbf{r}) = \mathbf{C} \qquad \text{where } \mathbf{C} = \text{const.} \quad (1\text{-}113)$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B} \quad (1\text{-}114)$$

$$\nabla\cdot(u\mathbf{A}) = \mathbf{A}\cdot(\nabla u) + u(\nabla\cdot\mathbf{A}) \quad (1\text{-}115)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \quad (1\text{-}116)$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B} \quad (1\text{-}117)$$

$$\nabla\times(u\mathbf{A}) = (\nabla u)\times\mathbf{A} + u(\nabla\times\mathbf{A}) \quad (1\text{-}118)$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \quad (1\text{-}119)$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \quad (1\text{-}120)$$

where

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right) + \hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right) \quad (1\text{-}121)$$

VECTOR OPERATIONS

RECTANGULAR COORDINATES

$$\nabla u = \hat{\mathbf{x}}\frac{\partial u}{\partial x} + \hat{\mathbf{y}}\frac{\partial u}{\partial y} + \hat{\mathbf{z}}\frac{\partial u}{\partial z} \quad (1\text{-}37)$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (1\text{-}42)$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad (1\text{-}43)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad (1\text{-}46)$$

CYLINDRICAL COORDINATES

$$\nabla u = \hat{\boldsymbol{\rho}}\frac{\partial u}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \hat{\mathbf{z}}\frac{\partial u}{\partial z} \quad (1\text{-}85)$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z} \quad (1\text{-}87)$$

$$\nabla\times\mathbf{A} = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{\mathbf{z}}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\varphi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\varphi}\right] \quad (1\text{-}88)$$

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} \quad (1\text{-}89)$$

SPHERICAL COORDINATES

$$\nabla u = \hat{\mathbf{r}}\frac{\partial u}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi} \quad (1\text{-}101)$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} \quad (1\text{-}103)$$

$$\nabla\times\mathbf{A} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right] + \frac{\hat{\boldsymbol{\varphi}}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \quad (1\text{-}104)$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} \quad (1\text{-}105)$$

CONVERSION OF SYMBOLS IN EQUATIONS

Symbols representing essentially mechanical quantities (length, mass, time, force, work, energy, power, etc.) are not changed (nor are derivatives). To convert an equation written in the MKSA system to the corresponding one in the Gaussian system, replace the symbol listed under the column labeled MKSA by that listed under Gaussian. The entries can also be used to convert a Gaussian equation to an MKSA one by going from right to left in the table.

Quantity	MKSA	Gaussian
Capacitance	C	$4\pi\epsilon_0 C$
Charge	q	$(4\pi\epsilon_0)^{1/2} q$
Charge density	$\rho, (\sigma, \lambda)$	$(4\pi\epsilon_0)^{1/2} \rho, (\sigma, \lambda)$
Conductivity	σ	$4\pi\epsilon_0 \sigma$
Current	I	$(4\pi\epsilon_0)^{1/2} I$
Current density	$\mathbf{J}, (\mathbf{K})$	$(4\pi\epsilon_0)^{1/2} \mathbf{J}, (\mathbf{K})$
Dielectric constant	κ_e	ϵ
Dipole moment (electric)	$\mathbf{p}$	$(4\pi\epsilon_0)^{1/2} \mathbf{p}$
Dipole moment (magnetic)	$\mathbf{m}$	$(4\pi/\mu_0)^{1/2} \mathbf{m}$
Displacement	$\mathbf{D}$	$(\epsilon_0/4\pi)^{1/2} \mathbf{D}$
Electric field	$\mathbf{E}$	$(4\pi\epsilon_0)^{-1/2} \mathbf{E}$
Inductance	L	$(4\pi\epsilon_0)^{-1} L$
Magnetic field	$\mathbf{H}$	$(4\pi\mu_0)^{-1/2} \mathbf{H}$
Magnetic flux	Φ	$(\mu_0/4\pi)^{1/2} \Phi$
Magnetic induction	$\mathbf{B}$	$(\mu_0/4\pi)^{1/2} \mathbf{B}$
Magnetization	$\mathbf{M}$	$(4\pi/\mu_0)^{1/2} \mathbf{M}$
Permeability	μ	(1) $\kappa_m \mu_0$, then (2) $\kappa_m \to \mu$
Permeability (relative)	κ_m	μ
Permittivity	ϵ	(1) $\kappa_e \epsilon_0$, then (2) $\kappa_e \to \epsilon$
Polarization	$\mathbf{P}$	$(4\pi\epsilon_0)^{1/2} \mathbf{P}$
Resistance	R	$(4\pi\epsilon_0)^{-1} R$
Resistivity	ρ	$(4\pi\epsilon_0)^{-1} \rho$
Scalar potential	ϕ	$(4\pi\epsilon_0)^{-1/2} \phi$
Speed of light	$(\mu_0\epsilon_0)^{-1/2}$	c
Susceptibility	$\chi_e, (\chi_m)$	$4\pi\chi_e, (\chi_m)$
Vector potential	$\mathbf{A}$	$(\mu_0/4\pi)^{1/2} \mathbf{A}$

VECTOR INTEGRAL FORMULAS

$$\oint_S \mathbf{A} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{A}\, d\tau \quad \text{(Divergence theorem)} \qquad (1\text{-}59)$$

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} \quad \text{(Stokes' theorem)} \qquad (1\text{-}67)$$

$$\oint_S u\, d\mathbf{a} = \int_V \nabla u\, d\tau \qquad (1\text{-}122)$$

$$\oint_S \mathbf{A} \times d\mathbf{a} = -\int_V (\nabla \times \mathbf{A})\, d\tau \qquad (1\text{-}123)$$

$$\oint_C u\, d\mathbf{s} = -\int_S \nabla u \times d\mathbf{a} \qquad (1\text{-}124)$$

$$\oint_S u\mathbf{A} \cdot d\mathbf{a} = \int_V [\mathbf{A} \cdot (\nabla u) + u(\nabla \cdot \mathbf{A})]\, d\tau \qquad (1\text{-}127)$$

$$\oint_S \mathbf{B}(\mathbf{A} \cdot d\mathbf{a}) = \int_V [(\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B}(\nabla \cdot \mathbf{A})]\, d\tau \qquad (1\text{-}129)$$

FORMULAS INVOLVING RELATIVE COORDINATES

$$\frac{\partial f(\mathbf{R})}{\partial x} = -\frac{\partial f(\mathbf{R})}{\partial x'} \qquad (1\text{-}130)$$

$$\nabla f(\mathbf{R}) = -\nabla' f(\mathbf{R}) \qquad (1\text{-}132)$$

$$\nabla \cdot \mathbf{A}(\mathbf{R}) = -\nabla' \cdot \mathbf{A}(\mathbf{R}) \qquad (1\text{-}133)$$

$$\nabla \times \mathbf{A}(\mathbf{R}) = -\nabla' \times \mathbf{A}(\mathbf{R}) \qquad (1\text{-}134)$$

$$\nabla^2 f(\mathbf{R}) = \nabla'^2 f(\mathbf{R}) \qquad (1\text{-}135)$$

$$\nabla R = -\nabla' R = \frac{\mathbf{R}}{R} = \hat{\mathbf{R}} \qquad (1\text{-}137)$$

$$\nabla\left(\frac{1}{R}\right) = -\nabla'\left(\frac{1}{R}\right) = -\frac{\hat{\mathbf{R}}}{R^2} = -\frac{\mathbf{R}}{R^3} \qquad (1\text{-}141)$$

$$\nabla^2\left(\frac{1}{R}\right) = \nabla'^2\left(\frac{1}{R}\right) = 0 \quad (R \neq 0) \qquad (1\text{-}144)$$