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Electricity and Magnetism — Preliminary Exam — January 17, 1991

The time limit for this exam is 3 hours. Answer all 10 questions. Please start each problem on a new page and show all work. Please use mks units if at all possible.

1. (12 points) Consider two conducting spheres of radius a and b ($a < b$). The region between the spheres is filled with an inhomogeneous, linear dielectric with $\epsilon = \epsilon_0/(c - \alpha r)$ where c and α are constants. A charge q is placed on the inner sphere.

   a) Find the electric displacement vector $\vec{D}$ in the region between the spheres.

   b) Find the volume polarization charge density, ρ_p, in the dielectric.

   c) Show explicitly that the total polarization charge is zero.

2. (8 points) A sphere of radius a has a charge density that varies with distance r from the center according to $\rho = Ar^{-1/2}$ where A is a constant. Find the electric field everywhere.

3. (10 points) A homogeneous, isotropic sphere of conductivity g is subjected to a potential $\phi_0\cos\theta$ at all points on its surface. Here θ is the usual polar angle measured with respect to an axis through the center of the sphere. What is the current density $\vec{J}$ at all points inside the sphere?

4. (10 points) A parallel plate capacitor consists of two circular plates of area A with vacuum between them. It is connected to a battery of constant emf V_0. The plates are then slowly oscillated so that they remain parallel but the separation d between them varies as $d = d_0 + d_1\sin\omega t$.

   a) Find the magnetic field $\vec{H}$ between the plates.

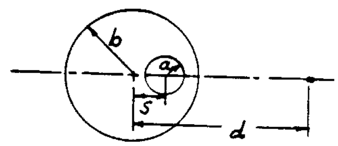
   b) Do the same for the case in which the battery is disconnected before the plates are oscillated.

5. (10 points) A plane electromagnetic wave travelling in a vacuum is given by $\vec{E} = E_0 \exp[i(\kappa z - \omega t)]\hat{\jmath}$ where E_0 is real. A circular loop of radius a, N turns, and resistance R is located with its center at the origin. The loop is oriented so that a diameter lies along the z-axis and the plane of the loop makes an angle θ with the y-axis.

   a) Find the emf induced in the loop.

   b) For part (a) we must assume that $a \ll \lambda$. Why?

6. (10 points) A cylindrical conductor of radius b contains a cylindrical hole of radius a; the axis of the hole is parallel to the axis of the conductor and a distance s away from it, as shown in the diagram. A current I flows in the conductor.



   a) Find the B-field in the hole on the diameter that coincides with a diameter of the conductor.

   b) Find the B-field at a point outside the conductor at a distance d from the axis and on the same diameter as in part (a).

7. (10 points) A circular disk rotates about its axis with angular velocity ω. The disk is made of metal with conductivity g, and its thickness is t. The rotating disk is placed between the pole faces of a magnet which produces a uniform magnetic field B over a small square area of size a^2 at the average distance r from the axis; B is perpendicular to the disk. Find the approximate torque on the disk.

8. (10 points) Suppose that the potential of a point charge is not the Coulomb potential but the Yukawa potential,

$$\phi = \frac{q e^{-\kappa r}}{4\pi\epsilon_0 r}.$$

   a) Find the electric field $\vec{E}$.

   b) Is Gauss' Law valid for this electric field?

   c) Is $\vec{\nabla} \times \vec{E} = 0$?

9. (10 points) A capacitance C in parallel with a resistance R has an impedance Z. Suppose that a capacitance C' in series with a resistance R' has the same impedance Z.

   a) Find the required values of C' and R' in terms of C and R for a given value of ω.

   b) Find the phase angle between the applied voltage and current.

10. (10 points) Consider a long solenoid of N/ℓ turns per unit length and radius R, such that the field inside is approximately uniform and the field outside is zero. A battery maintains a current I in the solenoid.

   a) From the magnetic energy, find the radial force on <u>one turn of the winding</u>, <u>per unit length of circumference</u>.

   b) Find the pressure inside the solenoid, and use this result to determine the relationship between the magnetic pressure, p_m, and the magnetic energy density u.

Solution to Laplace's equation in spherical polar coordinates:

$$\varphi_n = r^n P_n(\theta) \qquad \text{or} \qquad \varphi_n = r^{-(n+1)} P_n(\theta),$$

n	$P_n(\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta - 1)$
3	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

1. RECTANGULAR COORDINATES

$$\nabla\varphi = \mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z},$$

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

$$\nabla\times\mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right).$$

2. CYLINDRICAL COORDINATES

$$\nabla\varphi = \mathbf{a}_r\frac{\partial\varphi}{\partial r} + \mathbf{a}_\theta\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \mathbf{k}\frac{\partial\varphi}{\partial z},$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z},$$

$$\nabla\times\mathbf{F} = \mathbf{a}_r\left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right) + \mathbf{a}_\theta\left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) + \mathbf{k}\frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right).$$

3. SPHERICAL COORDINATES

$$\nabla\varphi = \mathbf{a}_r\frac{\partial\varphi}{\partial r} + \mathbf{a}_\theta\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \mathbf{a}_\phi\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi},$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi},$$

$$\nabla\times\mathbf{F} = \mathbf{a}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(F_\phi\sin\theta) - \frac{\partial F_\theta}{\partial\phi}\right] + \mathbf{a}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right] + \mathbf{a}_\phi\frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right].$$

Rectangular coordinates:

$$\nabla^2\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}.$$

Spherical coordinates:

$$\nabla^2\varphi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

Cylindrical coordinates:

$$\nabla^2\varphi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2}.$$

Table 1-1 Differential Vector Identities

$$
\begin{aligned}
\nabla\cdot\nabla\varphi &= \nabla^2\varphi \\
\nabla\cdot\nabla\times\mathbf{F} &= 0 \\
\nabla\times\nabla\varphi &= 0 \\
\nabla\times(\nabla\times\mathbf{F}) &= \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F} \\
\nabla(\varphi\psi) &= (\nabla\varphi)\psi + \varphi\nabla\psi \\
\nabla(\mathbf{F}\cdot\mathbf{G}) &= (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}\times(\nabla\times\mathbf{G}) + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{G}\times(\nabla\times\mathbf{F}) \\
\nabla\cdot(\varphi\mathbf{F}) &= (\nabla\varphi)\cdot\mathbf{F} + \varphi\nabla\cdot\mathbf{F} \\
\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= (\nabla\times\mathbf{F})\cdot\mathbf{G} - (\nabla\times\mathbf{G})\cdot\mathbf{F} \\
\nabla\times(\varphi\mathbf{F}) &= (\nabla\varphi)\times\mathbf{F} + \varphi\nabla\times\mathbf{F} \\
\nabla\times(\mathbf{F}\times\mathbf{G}) &= (\nabla\cdot\mathbf{G})\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}
\end{aligned}
$$