QFT Problem Set 4 - Due April 16

You should read chapters 24, 25, 26, 28. As usual, *problems* are for extra credit seekers, although everyone should look at them.

1. **Equivalent photon approximation** Consider the process in which electrons of very high energy scatter from a target. In leading order in α, the electron is connected to the target by one photon propagator. If the initial and final energies of the electron are E and E' the photon will carry momentum q such that $q^2 = -2EE'(1-\cos\theta)$. In the limit of forward scattering, whatever the energy loss, the photon momentum approaches $q^2 = 0$; thus the reaction is highly peaked in the forward direction. It is tempting to guess that, in this limit, the virtual photon becomes a real photon. Let us investigate in what sense that is true.

   (a) The matrix element for the scattering pro cess can b e written as

   $$M = -ie\bar{u}(p')\gamma^\mu u(p)\frac{-ig_{\mu\nu}}{q^2}\hat{M}^\nu(q) \tag{1}$$

   where $\hat{M}^\nu$ represents the coupling of the virtual photon to the target. Let $q = (q^0, \vec{q})$ and define $\tilde{q} = (q^0, -\vec{q})$. Expand

   $$\bar{u}(p')\gamma^\mu u(p) = Aq^\mu + B\tilde{q}^\mu + C\epsilon_1^\mu + D\epsilon_2^\mu \tag{2}$$

   where ϵ_I are unit vectors transverse to $\vec{q}$. Dot this with q and show that B is at most of order θ^2, so we can ignore it henceforth. Why is the coefficient A irrelevant?

   (b) Working in the frame with $p = (E, 0, 0, E)$, compute

   $$\bar{u}(p')\gamma \cdot \epsilon_i u(p) \tag{3}$$

   explicitly using massless electrons, where $\bar{u}$ and u are spinors of definite helicity, and ϵ_i are unit vectors parallel and perpendicular to the plane of scattering. We only want to keep the terms of order θ. Note that for ϵ_1 in the plane of scattering, the small $\hat{3}$ component of ϵ also contributes.

   (c) Now write the expression for the electron scattering cross section, in terms of $|\hat{M}^\mu|^2$ and the integral over phase space on the target side. This expression must be integrated over the final electron momentum p'. The integral over p'^3 is an integral over the energy loss of the electron. Show that the integral over $p'_\perp$ diverges logarithmically as $p'_\perp$ or $\theta \to 0$.

   (d) The divergence as $\theta \to 0$ appears because we have ignored the electron mass everywhere. Show that reintroducing the electron mass in the expression

   $$q^2 = -2(EE' - pp'\cos\theta) + 2m^2 \tag{4}$$

   cuts of the divergence and gives a factor of $\log(s/m^2)$ in its place.

(e) Assembling all the factors, and assuming that the target cross sections are independent of photon polarization, show that the largest part of the electron-target cross section is given by considering the electron to be the source of a beam of real photons with energy distribution in $x = E_\gamma/E$:

$$N_\gamma(x)dx = \frac{dx}{x}\frac{\alpha}{2\pi}(1+(1-x)^2)\log\frac{s}{m^2} \tag{5}$$

This is the Weizsacker-Williams equivalent photon approximation. It is a precursor to the theory of jets and partons in QCD. Note that it allows us to study photon-photon scattering using e^+e^- collisions.

2. **Rosenbluth Formula** Consider our formula for the effective photon-charged particle vertex

$$\bar{u}(p_2)\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p_q) \tag{6}$$

where $q = p_2 - p_1$. If the fermion is strongly interacting, e.g. the proton, then $F_i(q^2)$ can be quite non-trivial (hard or impossible to calculate) functions determined by the interal structure of the object. However, these form factors can be determined experimentally. Show that if an electron scatters off a proton (at rest) with energy $E \gg m_e$ then we find an elastic cross section

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2\left[(F_1^2 - \frac{q^2}{4m^2}F_2^2)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2E^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)\sin^4\frac{\theta}{2}} \tag{7}$$

where θ is the lab-frame scattering angle and F_i are evaluated at the associated q^2.

3. **Yukawa Couplings in QED** Consider QED with a Dirac spinor field ψ supplemented by a Yukawa interaction

$$L_{yukawa} = \lambda\phi\bar{\psi}\psi \tag{8}$$

where ϕ is a real scalar field. Verify that the contribution to Z_1 from the vertex diagram with a virtual ϕ equals the contribution to Z_2 from the diagram with a virtual ϕ. Now consider the renormalization of the $\phi\bar{\psi}\psi$ vertex. Show that the rescaling of this vertex is not canceled by the correction to Z_2 (you only need to compute short-distance divergences, or $1/\epsilon$ poles in dim reg, in order to find this).

4. **Lie Algebras and $SU(3)$** From the Gell-Mann matrices (on page 485 of the book, for example), work out the weights of the fundamental representation of $SU(3)$. Now work out the roots and draw the root lattice.

   Show that the three Gell-Mann matrices $\lambda^2, \lambda^5, \lambda^7$ generate an $SU(2)$ sub-algebra of $SU(3)$. How does the fundamental of $SU(3)$ transform according to this $SU(2)$? How does the adjoint of $SU(3)$ transform according to this $SU(2)$, or in other words, what irreps of $SU(2)$ does it break up into?

5. **Book Problems** 24.1, *26.3*