QFT Problem Set 7 - Due Dec. 8

You should read chapters 18, 19, 21, 22, and (especially) 23 of the book, focusing general principles. For some history you might enjoy reading Weinberg's (http://arxiv.org/pdf/0908.1964.pdf). As usual, *problems* are for theorists and extra credit seekers, although everyone should look at them.

1. **Sum Rule for Kallen-Lehmann** Show that the Kallen-Lehmann spectral function $\rho(\mu^2)$ obeys a sum rule

$$\int d\mu^2 \rho(\mu^2) = 1 \tag{1}$$

when we use it to represent the 2-pt function of a canonically normalized exact, Heisenberg picture scalar field with

$$[\dot{\phi}(t,x), \phi(t,y)] = -i\delta^3(\vec{x}-\vec{y}) \tag{2}$$

so that ϕ has the same canonical commutation relation as a free field (as it should).

2. **Renormalization and Symmetry** Consider a theory with two massless scalar fields in $3+1$ dimensions, with an interaction Lagrangian

$$\mathcal{L}_I = -\frac{g^2}{4!}(\phi_1^4 + \phi_2^4) - \frac{2\lambda}{4!}\phi_1^2\phi_2^2 \tag{3}$$

Note that when $g^2 = \lambda$ there is an $O(2)$ symmetry under rotations in the (ϕ_1, ϕ_2) plane. Now renormalize this theory at one-loop. Are the renormalized mass terms symmetric at one-loop? Compute the renormalization flow equations for the couplings and note that the symmetric limit is preserved under flow. Furthermore, show that if the initial values satisfy $\lambda/g^2 < 3$ then the theory becomes $O(2)$ symmetric in the low-energy limit... so one can have a low-energy symmetry that 'emerges' from a less symmetric theory.

3. **Asymptotic Behavior of Diagrams in ϕ^4 Theory** Compute the leading term in the S-Matrix elements for 2-to-2 scattering in $\frac{\lambda}{4!}\phi^4$ theory in the limit $s \to \infty$, with t fixed. Ignore all masses on internal lines, and only keep the mass non-zero as a useful low-energy regulator where it's needed. Show that

$$i\mathcal{M}(s,t) \approx -i\lambda - i\frac{\lambda^2}{(4\pi)^2}\log s - i\frac{5\lambda^3}{2(4\pi)^4}\log^2 s + \cdots \tag{4}$$

Note that by ignoring internal masses there are some nice simplifications in the Feynman parameter integrals.

4. **Old Fashioned Renormalizability** An idea that used to be sacred in QFT, but that is now viewed as pretty unimportant, is that of theories where one can cancel all short-distance (UV) divergences with a finite number of counterterms. Such theories were called 'renormalizable'; this classification was first understood by Dyson, and was alluded to in class.

Argue that at one-loop, the theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{5}$$

is renormalizable in $d \leq 4$ dimensions, and that we only need counterterms for two of the three terms that appear in the Lagrangian. At what loop order do we need counter-terms for all three? Now consider a theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda_4}{4!}\phi^4 - \frac{\lambda_6}{6!}\phi^6 \tag{6}$$

What infinite set of counter-terms do we need to absorb the UV divergences in $d = 4$? What about the case where instead of ϕ^6 we add a

$$\frac{(\partial\phi)^4}{\Lambda^d} \tag{7}$$

interaction to the Lagrangian? What would be different if we added a ϕ^5 interaction?

Renormalizability was viewed as important because there was a (feared) loss of predictivity in the presence of an infinite number of counterterms. You should make sure you understand why this isn't a problem for predictivity at low energies, but why it does mean that the theory under discussion will be incomplete (ie it cannot be extrapolated to arbitrarily short distances).