QFT Problem Set 6 - Due Dec. 1

You should skim chapters 15 - 19, focusing on the material we've covered. As usual, *problems* are for extra credit seekers, although everyone should look at them.

1. **Book Problems** 15.3, 18.1, 19.1, *15.4*, *Appendix B.1*, *19.3*

2. **Integrating Out Charged Particles** Consider scalar QED, where the scalar has mass M. Are there any tree level Feynman diagrams for scattering of photons into photons (2-to-2)? What diagrams contribute to this process at one-loop? Estimate the leading result when the photons in the scattering process have momentum much less than the scale set by M. Write down the corresponding effective operator(s) that contribute to leading order in the Lagrangian. With M set equal to the electron mass, are you surprised that optical photons don't scatter off of each other?

3. **Kadanoff Block Spins and Renormalization Flow** Here we will explicitly derive a very simple example of renormalization flow. Consider a 1-d Ising model with partition function

$$Z = \sum_{S_i = \pm 1} \exp\left[\sum_i L(i, i+1)\right] \quad \text{where} \quad L(i,j) = K S_i S_j + \frac{1}{2} h (S_i + S_j) \tag{1}$$

where K and h are constants, and S_i are spin variables that can take values $S_i = \pm 1$. Since we sum over all possible states to compute the partition function, we sum over all 2^N possibilities for the N different S_i. You can assume that the chain of spins forms a circle (so we have periodic boundary conditions).

Note that the exponent of the partition function plays exactly the same role as the action for a QFT. When we discuss path integrals you will learn that statistical mechanics is basically just QFT with $t \to it$.

Let us now integrate out the even numbered spins, to get a new action for the odd spins. This is a concrete way to integrate out short distance modes and derive an effective theory for the long-distance modes.

   (a) Show that if we sum over the even spins, we are left with a new effective Lagrangian for the odd spins

$$\tilde{L}(i, i+2) = \frac{1}{2} h (S_i + S_{i+2}) + \log\left(2 \cosh\left[K(S_i + S_{i+2}) + h\right]\right) \tag{2}$$

   Expand this to linear order in the external magnetic field h.

   (b) Show that

$$\begin{aligned} (S_i + S_j)^{2n} &= 2^{2n-1}(1 + S_i S_j) \qquad (3) \\ (S_i + S_j)^{2n+1} &= 2^{2n}(S_i + S_j) \qquad (4) \end{aligned}$$

(c) Use this to write the new action $\tilde{L}$ at linear order in h as

$$\tilde{L}(i, i+2) = K' S_i S_{i+2} + \frac{1}{2} h' (S_i + S_{i+2}) \tag{5}$$

You should find that

$$\begin{aligned} K' &= \frac{1}{2} \log \cosh(2K) & (6) \\ h' &= h[1 + \tanh(2K)] + \mathcal{O}(h^2) & (7) \end{aligned}$$

We have derived the RG equations for the evolution of the lagrangian parameters K and h as we zoom out to larger and larger distances. Note that in this theory we were very lucky that the renormalized Lagrangian took the same form as the original Lagrangian – in general many other terms will be generated. Polchinski's classic "Renormalization and Effective Lagrangians" (Nucl.Phys. B231 (1984) 269-295) applies this in a QFT context.