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THE JOINT SERVICES OF MONEY AND CREDIT

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Johns Hopkins Institute for Applied Economics,
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The Joint Services of Money and Credit

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About the Series

The *Studies in Applied Economics* series is under the general direction of Professor Steve H. Hanke, Co-Director of the Institute for Applied Economics, Global Health and Study of Business Enterprise at Johns Hopkins University (hanke@jhu.edu).

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Summary

While credit cards provide transaction services, as do currency and demand deposits, credit cards have never been included in measures of the money supply. The reason is accounting conventions, which do not permit adding liabilities, such as credit card balances, to assets, such as money. But economic aggregation theory and index number theory are based on microeconomic theory, not accounting, and measure service flows. We derive theory needed to measure the joint services of credit cards and money. The underlying assumption is that credit card services are not weakly separable from the services of monetary assets. Carried forward rotating balances are not included, since they were used for transactions services in prior periods. The theory is developed for the representative consumer, who pays interest for the services of credit cards during the period used for transactions. In the transmission mechanism of central bank policy, our results raise potentially fundamental questions about the traditional dichotomy between money and some forms of short term credit, such as checkable lines of credit. We do not explore those deeper issues in this paper, which focuses on measurement.

Keywords: credit cards, money, credit, aggregation theory, index number theory, Divisia index, risk, asset pricing.

JEL Classification Codes: C43, E01, E3, E40, E41, E51, E52, E58.

1. Introduction

Most models of the monetary policy transmission mechanism operate through interest rates, and often involve a monetary or credit channel, but not both. See, e.g., Bernanke and Blinder (1988) and Mishkin (1996). In addition, there are multiple versions of each mechanism, usually implying different roles for interest rates during the economy's adjustment to central bank policy actions. However, there is a more fundamental reason for separating money from credit. While money is an asset, credit is a liability. In accounting conventions, assets and liabilities are not added together. But aggregation theory and economic index number theory are based on microeconomic theory, not accounting conventions. Economic aggregates

measure service flows. To the degree that money and some forms of credit produce joint services, those services can be aggregated.

A particularly conspicuous example is credit card services, which are directly involved in transactions and contribute to the economy's liquidity in ways not dissimilar to those of money.[1] While this paper focuses on aggregation over monetary and credit card services, the basic principles could be relevant to some other forms of short term credit that contribute to the economy's liquidity services, such as checkable lines of credit.

While money is both an asset and part of wealth, credit cards are neither. Hence credit cards are not money. To the degree that monetary policy operates through a wealth effect (Pigou effect), as advocated by Milton Friedman, credit cards do not play a role. But to the degree that the flow of monetary services is relevant to the economy, as through the demand for monetary services or as an indicator measure, the omission of credit card services from "money" measures induces a loss of information. For example, Duca and Whitesell (1995) showed that a higher probability of credit card ownership was correlated with lower holdings of monetary transactions balances. Clearly credit card services are a substitute for the services of monetary transactions balances, and perhaps to a much higher degree than the services of many of the assets included in traditional monetary aggregates, such as the services of nonnegotiable certificates of deposit.

In this seminal paper, we use strongly simplifying assumptions. We assume credit cards are used only to purchase consumer goods. All purchases are made at the beginning of periods, and payments for purchases are either by credit cards or money. Credit card purchases are fully repaid to the credit card company at the end of the period, plus interest charged by the credit card company. The assumption of repayment of all credit card debt at the end of each period is only for expository convenience. The extension to revolving credit, including credit card debt carried

[1] We are indebted to Apostolos Serletis for his suggestion of this topic for research. His suggestion is contained in his presentation as discussant of Barnett's Presidential Address at the Inaugural Conference of the Society for Economic Measurement at the University of Chicago, August 18-20, 2014. The slides for Serletis's discussion can be found online at http://sem.society.cmu.edu/conference1.html.

forward to future periods, is provided in section (2.1). After aggregation over consumers, the expected interest rate paid by the representative credit card holder can be very high. Future research is planned to disaggregate to heterogeneous agents, including consumers who repay soon enough to owe no interest.

To reflect the fact that money and credit cards provide services, such as liquidity and transactions services, money and credit are entered into a derived utility function, in accordance with Arrow and Hahn's (1971) proof.[2] The derived utility function absorbs constraints reflecting the explicit motives for using money and credit card services. Since this paper is about measurement, we need only assume the existence of such motives. In the context of this research, we have no need to work backwards to reveal the explicit motives. As has been shown repeatedly, any of those motives, including the highly relevant transactions motive, are consistent with existence of a derived utility function absorbing the motive.[3]

2. Intertemporal Allocation

We begin by defining the variables in the risk neutral case:

$\mathbf{x}_s$ = vector of per capita (planned) consumptions of N goods and services (including those of durables) during period s.

[2] Our research in this paper is not dependent upon the simple decision problem we use for derivation and illustration. In the case of monetary aggregation, Barnett (1987) proved that the same aggregator functions and index numbers apply, regardless of whether the initial model has money in the utility function or production function, so long as there is intertemporal separability of structure and separability of components over which aggregation occurs. That result is equally as applicable to our current results with augmented aggregation over monetary asset and credit card services. While this paper uses economic index number theory, it should be observed that there also exists a statistical approach to index number theory. That approach produces the same results, with the Divisia index interpreted to be the Divisia mean using expenditure shares as probability. See Barnett and Serletis (1990).

[3] The aggregator function is the derived function that always exists, if monetary and credit card services have positive value in equilibrium. See, e.g., Samuelson (1948), Arrow and Hahn (1971), Stanley Fischer (1974), Phlips and Spinnewyn (1982), Quirk and Saposnik (1968), and Poterba and Rotemberg (1987). Analogously Feenstra (1986, p. 271) demonstrated "a functional equivalence between using real balances as an argument of the utility function and entering money into liquidity costs which appear in the budget constraints." The converse mapping from money and credit in the utility function approach back to the explicit motive is not unique, but in this paper we are not seeking to identify the explicit motives for holding money or credit card balances.

$\mathbf{p}_s$ = vector of goods and services expected prices, and of durable goods expected rental prices during period s.

m_{is} = planned per capita real balances of monetary asset i during period s $(i = 1,2, \dots, n)$.

c_{js} = planned per capita real balances of credit card type j during period s $(j = 1,2, \dots, k)$.

r_{is} = expected nominal holding period yield (including capital gains and losses) on monetary asset i during period s $(i = 1,2, \dots, n)$.

e_{js} = expected interest rate on credit card type j during period s $(j = 1,2, \dots, k)$.

A_s = planned per capita real holdings of the benchmark asset during period s.

R_s = expected (one-period holding) yield on the benchmark asset during period s.

L_s = per capita labor supply during period s.

w_s = expected wage rate during period s.

The benchmark asset is defined to provide no services other than its expected yield, R_s, which motivates holding of the asset solely as a means of accumulating wealth. As a result, R_s is the maximum expected holding period yield available to consumers in the economy in period *s*. The benchmark asset is held to transfer wealth by consumers between multiperiod planning horizons, rather than to provide liquidity or other services.

The expected interest rate, e_{js}, can be explicit or implicit, and applies to the aggregated representative consumer. For example, an implicit part of that interest rate could be in the form of an increased price of the goods purchased or in the form of a periodic service fee or membership fee. The fact that many retailers do not offer discounts for cash is somewhat puzzling and might change in the future. Nevertheless, the expected rate of return to credit card companies, e_{js}, aggregated over consumers, tends to be very high, far exceeding R_s , even after substantial losses from fraud.

We let u_t be the representative consumer's current intertemporal utility function at time t over the T-period planning horizon. We assume that u_t is weakly separable in each period's consumption of goods and monetary assets, so that u_t can be written in the form

$$\begin{aligned} u_t &= u_t(\boldsymbol{m}_t, \dots, \boldsymbol{m}_{t+T};\ \boldsymbol{c}_t, \dots, \boldsymbol{c}_{t+T};\ \boldsymbol{x}_t, \dots, \boldsymbol{x}_{t+T};\ A_{t+T}) \\ &= U_t(v(\boldsymbol{m}_t, \boldsymbol{c}_t), v_{t+1}(\boldsymbol{m}_{t+1}, \boldsymbol{c}_{t+1}), \dots, v_{t+T}(\boldsymbol{m}_{t+T}, \boldsymbol{c}_{t+T}); \\ &\quad V(\boldsymbol{x}_t), V_{t+1}(\boldsymbol{x}_{t+1}), \dots, V_{t+T}(\boldsymbol{x}_{t+T}); A_{t+T}), \end{aligned} \tag{1}$$

for some monotonically increasing, linearly homogeneous, strictly quasiconcave functions, $v, v_{t+1}, \dots, v_{t+T}, V, V_{t+1}, \dots, V_{t+T}$. The function U_t also is monotonically increasing, but not necessarily linearly homogeneous.

Dual to the functions, V and V_s $(s = t+1, \dots, t+T)$, there exist current and planned true cost of living indexes, $p_t^* = p(\boldsymbol{p}_t)$ and $p_s^* = p_s^*(\boldsymbol{p}_s)$ $(s = t+1, \dots, t+T)$. Those indexes, which are the consumer goods unit cost functions, will be used to deflate all nominal quantities to real quantities, as in the definitions of m_{is}, c_{js} and A_s above.

Assuming replanning at each t, we write the consumer's decision problem during each period s $(t \le s \le t+T)$ within his planning horizon to be to choose $(\boldsymbol{m}_t, \dots, \boldsymbol{m}_{t+T};\ \boldsymbol{c}_t, \dots, \boldsymbol{c}_{t+T};\ \boldsymbol{x}_t, \dots, \boldsymbol{x}_{t+T};\ A_{t+T}) \ge \mathbf{0}$ to

$$\max u_t(\boldsymbol{m}_t, \dots, \boldsymbol{m}_{t+T};\ \boldsymbol{c}_t, \dots, \boldsymbol{c}_{t+T};\ \boldsymbol{x}_t, \dots, \boldsymbol{x}_{t+T};\ A_{t+T}),$$

subject to

$$\begin{aligned} \mathbf{p}_s'\mathbf{x}_s = w_s L_s &+ \sum_{i=1}^{n}\left[(1 + r_{i,s-1})p_{s-1}^* m_{i,s-1} - p_s^* m_{is}\right] \\ &+ \sum_{j=1}^{k}\left[p_s^* c_{js} - (1 + e_{j,s-1})p_{s-1}^* c_{j,s-1}\right] \\ &+ \left[(1 + R_{s-1})p_{s-1}^* A_{s-1} - p_s^* A_s\right]. \end{aligned} \tag{2}$$

Let

$$\rho_s = \begin{cases} 1, & if\ s = t, \\ \prod_{u=t}^{s-1}(1 + R_u), & if\ t+1 \le s \le t+T. \end{cases} \quad (3)$$

Equation (2) is a flow of funds identity, with the right hand side being funds available to purchase consumer good during period *s*. On the right hand side, the first term is labor income. The second term is funds absorbed or released by rolling over the monetary assets portfolio, as explained in Barnett (1980). The third term is particularly important to this paper. That term measures credit card debt accumulated during period *s* from purchases of consumer goods, minus the cost of paying off last period's credit card debt plus interest. The fourth term is funds absorbed or released by rolling over the stock of the benchmark asset, as explained in Barnett (1980).

We now derive the implied Fisherine discounted wealth constraint. The derivation procedure involves recursively substituting each flow of funds identity into the previous one, working backwards in time, as explained in Barnett (1980). The result is the following wealth constraint at time *t*:

$$\sum_{s=t}^{t+T}\left(\frac{\mathbf{p}_s'}{\rho_s}\right)\mathbf{x}_s+\sum_{s=t}^{t+T}\sum_{i=1}^{n}\left[\frac{p_s^*}{\rho_s}-\frac{p_s^*(1+r_{i,s})}{\rho_{s+1}}\right]m_{i,s}+\sum_{i=1}^{n}\frac{p_{t+T}^*(1+r_{i,t+T})}{\rho_{t+T+1}}m_{i,t+T}+\frac{p_{t+T}^*}{\rho_{t+T}}A_{t+T}$$
$$+\sum_{s=t}^{t+T}\sum_{j=1}^{k}\left[\frac{p_s^*(1+e_{js})}{\rho_{s+1}}-\frac{p_s^*}{\rho_s}\right]c_{j,s}$$
$$=\sum_{s=t}^{t+T}\left(\frac{w_s}{\rho_s}\right)L_s$$
$$+\sum_{i=1}^{n}(1+r_{i,t-1})p_{t-1}^*m_{i,t-1}+(1+R_{t-1})A_{t-1}p_{t-1}^*$$
$$+\sum_{j=1}^{k}\frac{p_{t+T}^*(1+e_{j,t+T})}{\rho_{t+T+1}}c_{j,t+T}-\sum_{j=1}^{k}(1+e_{j,t-1})p_{t-1}^*c_{j,t-1}. \qquad (4)$$

It is important to understand that (4) is directly derived from (2) without any additional assumptions. As in Barnett (1978, 1980), we see immediately that the nominal user cost (equivalent rental price) of monetary asset holding m_{is} $(i=1,2,\dots,n)$ is

$$\pi_{is}=\frac{p_s^*}{\rho_s}-\frac{p_s^*(1+r_{is})}{\rho_{s+1}}.$$

So the current nominal user cost price, π_{it}, of m_{it} reduces to

$$\pi_{it}=\frac{p_t^*(R_t-r_{it})}{1+R_t}. \qquad (5)$$

Likewise, the nominal user cost (equivalent rental price) of credit card service c_{js} $(j=1,2,\dots,k)$ is

$$\tilde{\pi}_{jt}=\frac{p_s^*(1+e_{js})}{\rho_{s+1}}-\frac{p_s^*}{\rho_s}.$$

Finally the current period nominal user cost, $\tilde{\pi}_{jt}$, of c_{jt} reduces to

$$\tilde{\pi}_{jt} = \frac{p_t^*(1+e_{jt})}{1+R_t} - p_t^* \tag{6}$$

$$= \frac{p_t^*(e_{jt}-R_t)}{1+R_t}. \tag{7}$$

Equation (7) is a new result central to most that follows in this paper.[4] The corresponding real user costs are

$$\pi_{js}^* = \frac{\pi_{is}}{p_s^*}$$

and

$$\tilde{\pi}_{js}^* = \frac{\tilde{\pi}_{jt}}{p_s^*}.$$

Equation (6) is particularly revealing. To consume the transactions services of credit card type *j*, the consumer borrows p_t^* dollars per unit of goods purchased at the start of the period during which the goods are consumed, but repays the credit card company $p_t^*(1+e_{jt})$ dollars at the end of the period. The lender will not provide that one period loan to the consumer unless $e_{jt} > R_t$, because of the ability of the lender to earn R_t without making the unsecured credit card loan. The assumption that consumers do not have access to higher expected yields than the benchmark rate does not apply to firms, such as credit card firms. Hence the user cost price in (7) is nonnegative.[5]

[4] The same user cost formula applies in the infinite planning horizon case, but the derivation is different. The derivation applicable in that case is in the Appendix.

[5] Our model is of the representative consumer, aggregated over all credit card holders. In an extension to heterogeneous agents, we would separate out consumers who repay the credit card company soon enough to avoid interest on the loan. That possibility could be viewed as a special case of our current model, in which the consumer repays immediately. In that special case, there is no discounting between purchase and repayment, and no interest is charged. The services of the credit card company become a free good with user cost price of zero. The credit card debt then

Equivalently, equation (7) can be understood in terms of the delay between the goods purchase date and the date of repayment of the loan to the credit card company. During the one period delay, the consumer can invest the cost of the goods purchase at rate of return R_t. Hence the net real cost to the consumer of the credit card loan, per dollar borrowed, is $e_{jt} - R_t$. Multiplication by the true cost of living index in the numerator of (7) converts to nominal dollars and division by $1 + R_t$ discounts to present value within the time period.

2.1. Extension to Revolving Credit

There are two approaches to extending the above results to the case of revolving credit, which need not be paid off at the end of each period. The difference between the two methods depends upon the definition of c_{js}. One method defines c_{js} to be total debt balances in the credit card account. The other preferable method defines c_{js} to be those credit card balances used for purchases during period s. Under our simplifying assumption that credit card debt is fully paid off each period, the two approaches become identical.

Method 1: If c_{js} is defined to be total debt balances in the credit card account, all of the theory in this paper would be unchanged, but the interpretation of inclusion of credit card debt in the utility function would be altered in a somewhat disturbing manner. Under our assumption that credit card debt is fully paid off each period, all credit card balances produce transaction services each period. Without that hypothesis, the model under Method 1 would imply that total balances of credit card debt produce services, including balances carried forward from prior period's purchases. Since those carried forward balances provided transactions services in previous periods, keeping those balances in the utility function for the current period would imply existence of a different kind of services.

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disappears from the flow of funds equation, (2), since the credit cards provide no net services to the economy, and serve as instantaneous intermediaries in payment of goods purchased with money. Section 2.1 below considers more explicitly such extensions.

Method 2: The alternative method would provide the straightforward extension of our results to the case of rotating credit, with only current period credit card purchases providing transactions services. While theoretically preferable to Method 1, this approach has heavier data requirements. By this method, c_{js} is redefined as follows:

c_{js} = those planned per capita real balances of credit card type j used for transactions during period s $(j = 1,2, \dots, k)$.

Under this definition, total credit card balances could exceed c_{js}. The rotating balances, z_{js}, from previous periods not used for transactions this period would add a flow of funds term to the constraint, (2), but not appear in the utility function. All resulting aggregates, results, and theory below would be unchanged. But to implement that extension empirically, we would need data on total credit card transactions each period, c_{js}, not just the total balances in the accounts, $c_{js} + z_{js}$.[6]

To see this more clearly, rewrite equation (2) as

$$\begin{aligned}\mathbf{p}_s'\mathbf{x}_s = w_s L_s &+ \sum_{i=1}^{n}\left[(1 + r_{i,s-1})p^*_{s-1}m_{i,s-1} - p^*_s m_{is}\right] \\ &+ \sum_{j=1}^{k}\left[p^*_s y_{js} - (1 + e_{j,s-1})p^*_{s-1}y_{j,s-1}\right] \\ &+ \left[(1 + R_{s-1})p^*_{s-1}A_{s-1} - p^*_s A_s\right].\end{aligned} \tag{8a}$$

where , $y_{js} = c_{js} + z_{js}$. Clearly that equation then becomes:

[6] Credit card companies provide a line of credit to consumers, with interest and any late payments added after the due date. New purchases are added as debt to the balance after the due date has passed. Many consumers having balances, z_{js}, pay only the "minimum payment" due. That decision avoids a late charge, but adds the unpaid balance to the stock of debt and boosts the interest due. Depending upon the procedure for aggregating over consumers, the interest rate on c_{js} could be different from the interest rate on z_{js}, with the former interest rate being the one that should be used in our user cost formula.

$$\mathbf{p}_s'\mathbf{x}_s = w_s L_s + \sum_{i=1}^{n}\left[(1+r_{i,s-1})p_{s-1}^* m_{i,s-1} - p_s^* m_{is}\right]$$
$$+ \sum_{j=1}^{k}\left[p_s^* c_{js} - (1+e_{j,s-1})p_{s-1}^* c_{j,s-1}\right] \qquad (8b)$$
$$+ \sum_{j=1}^{k}\left[p_s^* z_{js} - (1+e_{j,s-1})p_{s-1}^* z_{j,s-1}\right]$$
$$+ \left[(1+R_{s-1})p_{s-1}^* A_{s-1} - p_s^* A_s\right].$$

The third term on the right side of equation (8a) is easily interpreted as the net increase in credit card debt between the two periods minus interest paid on last period's credit card debt. In equation (8b), the third term on the right side is specific to current period credit card purchases, while the fourth term is not relevant to the rest of our results, since z_{js} is not in the utility function. Hence z_{js} is not relevant to the user cost prices, conditional decisions, or aggregates in the rest of this paper. In short, equation (2) remains relevant under either the Method 1 or Method 2 interpretation, since the additional term introduced into (8b) by Method 2 plays no role in the rest of the analysis.

While Method 2 is preferable on theoretical grounds, the growth rates of the resulting aggregates might be similar under the two methods, since growth rate variations are likely to be dominated by the volatility of current transactions balances, rather than the smoother carried forward balances. If that proves not to be the case, and if data on carried forward credit card debt are not available, the best alternative might be to model that carried forward amount to be filtered out of the total.

3. Conditional Current Period Allocation

We define $\mathcal{J}_t^*$ to be real, and $\mathcal{J}_t$ nominal, expenditure on augmented monetary services --- augmented to include the services of credit card charges. The

assumptions on homogeneous blockwise weak separability of the intertemporal utility function, (1), are sufficient for consistent two-stage budgeting. See Green (1964, theorem 4). In the first stage, the consumer selects real expenditure on augmented monetary services, $\mathcal{J}_t^*$, and on aggregate consumer goods for each period within the planning horizon, along with terminal benchmark asset holdings, A_{t+T}.

In the second stage, $\mathcal{J}_t^*$ is allocated over demands for the current period services of monetary assets and credit cards. That decision is to select $\mathbf{m}_t$ and $\mathbf{c}_t$ to

$$\max v(\mathbf{m}_t, \mathbf{c}_t), \tag{9}$$

subject to

$$\boldsymbol{\pi}^{*\prime}{}_t \mathbf{m}_t + \tilde{\boldsymbol{\pi}}^{*\prime}{}_t \mathbf{c}_t = \mathcal{J}_t^*, \tag{10}$$

where $\mathcal{J}_t^*$ is expenditure on augmented monetary services allocated to the current period in the consumer's first-stage decision.

4. Aggregation Theory

The exact quantity aggregate is the level of the indirect utility produced by solving problem ((9),(10)):

$$\mathcal{M}_t = \max \{v(\mathbf{m}_t, \mathbf{c}_t) : \boldsymbol{\pi}_t' \mathbf{m}_t + \tilde{\boldsymbol{\pi}}_t' \mathbf{c}_t = \mathcal{J}_t\} \tag{11}$$

$$= \max \{v(\mathbf{m}_t, \mathbf{c}_t) : \boldsymbol{\pi}^{*\prime}{}_t \mathbf{m}_t + \tilde{\boldsymbol{\pi}}^{*\prime}{}_t \mathbf{c}_t = \mathcal{J}_t^*\},$$

where we define $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t) = v(\mathbf{m}_t, \mathbf{c}_t)$ to be the "augmented monetary aggregate" --- augmented to aggregate jointly over the services of money and credit. The category utility function v is the aggregator function we assume to be linearly

homogeneous in this section. Dual to any exact quantity aggregate, there exists a unique price aggregate, aggregating over the prices of the goods or services. Hence there must exist an exact nominal price aggregate over the user costs $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$. As shown in Barnett (1980,1987), the consumer behaves relative to the dual pair of exact monetary quantity and price aggregates as if they were the quantity and price of an elementary good. The same result applies to our augmented monetary quantity and dual user cost aggregates.

One of the properties that an exact dual pair of price and quantity aggregates satisfies is Fisher's factor reversal test, which states that the product of an exact quantity aggregate and its dual exact price aggregate must equal actual expenditure on the components. Hence, if $\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$ is the exact user cost aggregate dual to $\mathcal{M}_t$, then $\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$ must satisfy

$$\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) = \frac{\mathcal{J}_t}{\mathcal{M}_t}. \tag{12}$$

Since (12) produces a unique solution for $\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$, we could use (12) to define the price dual to $\mathcal{M}_t$. In addition, if we replace $\mathcal{M}_t$ by the indirect utility function defined by (11) and use the linear homogeneity of v, we can show that $\Pi = \Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$ defined by (12) does indeed depend only upon $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$, and not upon $(\mathbf{m}_t, \mathbf{c}_t)$ or $\mathcal{J}_t$. See Barnett (1987) for a version of the proof in the case of monetary assets. The conclusion produced by that proof can be written in the form

$$\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) = [\max_{(\mathbf{m}_t, \mathbf{c}_t)} \{v(\mathbf{m}_t, \mathbf{c}_t) : \boldsymbol{\pi}_t' \mathbf{m}_t + \tilde{\boldsymbol{\pi}}_t' \mathbf{c}_t = 1\}]^{-1}, \tag{13}$$

which clearly depends only upon $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$.

Although (13) provides a valid definition of Π, there also exists a direct definition that is more informative and often more useful. The direct definition depends upon the cost function E, defined by

$$E(v_0, \boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) = \min_{(\mathbf{m}_t, \mathbf{c}_t)} \{\boldsymbol{\pi}_t' \mathbf{m}_t + \tilde{\boldsymbol{\pi}}_t' \mathbf{c}_t : v(\mathbf{m}_t, \mathbf{c}_t) = v_0\},$$

which equivalently can be acquired by solving the indirect utility function equation (11) for $\mathcal{J}_t$ as a function of $\mathcal{M}_t = v(\mathbf{m}_t, \mathbf{c}_t)$ and $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$. Under our linear homogeneity assumption on v, it can be proved that

$$\begin{aligned} \Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) &= E(1, \boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) \\ &= \min_{(\mathbf{m}_t, \mathbf{c}_t)} \{\boldsymbol{\pi}_t' \mathbf{m}_t + \tilde{\boldsymbol{\pi}}_t' \mathbf{c}_t : v(\mathbf{m}_t, \mathbf{c}_t) = 1\}, \end{aligned} \tag{14}$$

which is often called the unit cost or price function.

The unit cost function is the minimum cost of attaining unit utility level for $v(\mathbf{m}_t, \mathbf{c}_t)$ at given user cost prices $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$. Clearly, (14) depends only upon $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$. Hence by (12) and (14), we see that $\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) = {}^{\mathcal{J}_t}/_{\mathcal{M}_t} = E(1, \boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$.

5. Preference Structure over Financial Assets

5.1. Blocking of the Utility Function

While our primary objective is to provide the theory relevant to joint aggregation over monetary and credit card services, subaggregation separately over monetary asset services and credit card services can be nested consistently within the joint aggregates. The required assumption is blockwise weak separability of money and credit within the joint aggregator function. In particular, we would then assume the existence of functions $\tilde{\upsilon}$, g_1, g_2, such that

$$v(\mathbf{m}_t, \mathbf{c}_t) = \tilde{\upsilon}\big(g_1(\boldsymbol{m}_t), g_2(\boldsymbol{c}_t)\big), \tag{15}$$

with the functions g_1 and g_2 being linearly homogeneous, increasing, and quasiconcave.

We have nested weakly separable blocks within weakly separable blocks to establish a fully nested utility tree. As a result, an internally consistent multi-stage budgeting procedure exists, such that the structured utility function defines the quantity aggregate at each stage, with duality theory defining the corresponding user cost price aggregates.

In the next section we elaborate on the multi-stage budgeting properties of decision ((9), (10)) and the implications for quantity and price aggregation.

5.2. Multi-stage Budgeting

Our assumptions on the properties of v are sufficient for a two-stage solution of the decision problem ((9), (10)), subsequent to the two-stage intertemporal solution that produced ((9),(10)). The subsequent two-stage decision is exactly nested within the former one.

Let $M_t = M(\mathbf{m}_t)$ be the exact aggregation-theoretic quantity aggregate over monetary assets, and let $C_t = C(\mathbf{c}_t)$ be the exact aggregation-theoretic quantity aggregate over credit card services. Let $\Pi_m^* = \Pi_m(\boldsymbol{\pi}_t^*)$ be the real user costs aggregate (unit cost function) dual to $M(\mathbf{m}_t)$, and let $\Pi_c^* = \Pi_c(\tilde{\boldsymbol{\pi}}_t^*)$ be the user costs aggregate dual to $C(\mathbf{c}_t)$. The first stage of the two-stage decision is to select M_t and C_t to solve

$$\max_{(\mathbf{m}_t, \mathbf{c}_t)} \tilde{\upsilon}(M_t, C_t) \tag{16}$$

subject to

$$\Pi_m^* M_t + \Pi_c^* C_t = \mathcal{J}_t^* .$$

From the solution to problem (16), the consumer determines aggregate real expenditure on monetary and credit card services, $\Pi_m^* M_t$ and $\Pi_c^* C_t$.

In the second stage, the consumer allocates $\Pi_m^* M_t$ over individual monetary assets, and allocates $\Pi_c^* C_t$ over services of individual types of credit cards. She does so by solving the decision problem:

$$\max_{\mathbf{m_t}} g_1(\boldsymbol{m}_t), \tag{17}$$

subject to

$$\boldsymbol{\pi}_t^{*\prime}\boldsymbol{m}_t = \Pi_m^* M_t.$$

Similarly, she solves

$$\max_{\mathbf{c_t}} g_2(\boldsymbol{c}_t), \tag{18}$$

subject to

$$\tilde{\boldsymbol{\pi}}_t^{*\prime}\boldsymbol{c}_t = \Pi_c^*\, C_t.$$

The optimized value of decision (17)'s objective function, $g_1(\boldsymbol{m}_t)$, is then the monetary aggregate, $M_t = M(\mathbf{m}_t)$, while the optimized value of decision (18)'s objective function, $g_2(\boldsymbol{c}_t)$, is the credit card services aggregate, $C_t = C(\mathbf{c}_t)$.

Hence,

$$M_t = \max\,\{g_1(\mathbf{m}_t)\colon \boldsymbol{\pi}_t^{*\prime}\mathbf{m}_t = \Pi_m^* M_t\} \tag{19}$$

and

$$C_t = \max\,\{g_2(\mathbf{c}_t)\colon \tilde{\boldsymbol{\pi}}_t^{*\prime}\mathbf{c}_t = \Pi_c^*\, C_t\}. \tag{20}$$

It then follows from (11) and (15) that the optimized values of the monetary and credit card quantity aggregates are related to the joint aggregate in the following manner:

$$\mathcal{M}_t = \tilde{\upsilon}(M_t, C_t). \tag{21}$$

6. The Divisia Index

We advocate using the Divisia index, in its Törnqvist (1936) discrete time version, to track $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$, as Barnett (1980) has previously advocated for tracking $M_t = M(\mathbf{m}_t)$. If there should be reason to track the credit card aggregate separately, the Törnqvist-Divisia index similarly could be used to track $C_t = C(\mathbf{c}_t)$. If there is reason to track all three individually, then after measuring M_t and C_t, the joint aggregate $\mathcal{M}_t$ could be tracked as a two-good Törnqvist-Divisia index using (21), rather as an aggregate over the $n + k$ disaggregated components, $(\mathbf{m}_t, \mathbf{c}_t)$. The aggregation theoretic procedure for selecting the $n + m$ component assets is described in Barnett (1982).

6.1. The Linearly Homogeneous Case

It is important to understand that the Divisia index (1925, 1926) in continuous time will track any aggregator function without error. To understand why, it is best to see the derivation. The following is a simplified version based on Barnett (2012, pp. 290-292), adapted for our augmented monetary aggregate, which aggregates jointly over money and credit card services. The derivation is equally as relevant to separate aggregation over monetary assets or credit cards, so long as the prices in the indexes are the corresponding user costs, ((5), (7)). Although Francois Divisia (1925, 1926) derived his consumer goods index as a line integral, the simplified approach below is mathematically equivalent to Divisia's original method.

At instant of continuous time, *t*, consider the quantity aggregator function, $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t) = v(\mathbf{m}_t, \mathbf{c}_t)$, with components $(\mathbf{m}_t, \mathbf{c}_t)$, having user cost prices $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$. Let $\mathbf{m}_t^a = (\mathbf{m}_t', \mathbf{c}_t')'$ and $\boldsymbol{\pi}_t^a = (\boldsymbol{\pi}_t', \tilde{\boldsymbol{\pi}}_t')'$. Take the total differential of $\mathcal{M}$ to get

$$d\mathcal{M}(\mathbf{m}_t^a) = \sum_{i=1}^{n+k} \frac{\partial \mathcal{M}}{\partial m_{it}^a} \, dm_{it}^a. \tag{22}$$

Since $\partial \mathcal{M} / \partial m_{it}$ contains the unknown parameters of the function $\mathcal{M}$, we replace each of those marginal utilities by $\lambda \pi_{it}^a = \partial \mathcal{M} / \partial m_{it}$ which is the first-order condition for expenditure constrained maximization of $\mathcal{M}$, where λ is the Lagrange multiplier, and π_{it}^a is the user-cost price of m_{it}^a at instant of time t.

We then get

$$\frac{d\mathcal{M}(\mathbf{m}_t^a)}{\lambda} = \sum_{i=1}^{n+k} \pi_{it}^a dm_{it}^a, \tag{23}$$

which has no unknown parameters on the right-hand side.

For a quantity aggregate to be useful, it must be linearly homogeneous. A case in which the correct growth rate of an aggregate is clearly obvious is the case in which all components are growing at the same rate. As required by linear homogeneity, we would expect the quantity aggregate would grow at that same rate. Hence we shall assume $\mathcal{M}$ to be linearly homogeneous.

Define $\Pi^a(\boldsymbol{\pi}_t^a)$ to be the dual price index satisfying Fisher's factor reversal test, $\Pi^a(\boldsymbol{\pi}_t^a)\mathcal{M}(\mathbf{m}_t^a) = \boldsymbol{\pi}_t^{a\prime}\mathbf{m}_t^a$. In other words, define $\Pi^a(\boldsymbol{\pi}_t^a)$ to equal $\boldsymbol{\pi}_t^{a\prime}\mathbf{m}_t^a / \mathcal{M}(\mathbf{m}_t^a)$, which can be shown to depend only upon $\boldsymbol{\pi}_t^a$, when $\mathcal{M}$ is linearly homogeneous. Then the following lemma holds.

Lemma 1: *Let λ be the Lagrange multiplier in the first order conditions for solving the constrained maximization ((9),(10)), and assume that v is linearly homogeneous. Then*

$$\lambda = \frac{1}{\Pi^a(\boldsymbol{\pi}_t^a)}$$

Proof: See Barnett (2012, p. 291). ■

From Equation (23), we therefore find the following:

$$\Pi^a(\boldsymbol{\pi}_t^a)d\mathcal{M}(\mathbf{m}_t^a) = \sum_{i=1}^{n+k} \pi_i^a dm_i^a. \tag{24}$$

Manipulating Equation (24) algebraically to convert to growth rate (log change) form, we find that

$$dlog\ \mathcal{M}(\mathbf{m}_t^a) = \sum_{i=1}^{n+k} \omega_{it}\ dlog\ m_i^a, \tag{25}$$

where $\omega_{it} = \pi_i^a m_i^a / \boldsymbol{\pi}_t^{a\prime} \mathbf{m}_t^a$ is the value share of m_i^a in total expenditure on the services of $\mathbf{m}_t^a$. Equation (25) is the Divisia index in growth rate form. In short, the growth rate of the Divisia index, $\mathcal{M}(\mathbf{m}_t^a)$, is the share weighted average of the growth rates of the components.[7] Notice that there were no assumptions at all in the derivation about the functional form of $\mathcal{M}$, other than existence (*i.e.*, weak separability within the structure of the economy) and linear homogeneity of the aggregator function.

If Divisia aggregation was previously used to aggregate separately over money and credit card services, then equation (25) can be replaced by a two-goods Divisia index aggregating over the two subaggregates, in accordance with equation (21).

6.2. The Nonlinearly Homogeneous Case

For expositional simplicity, we have presented the aggregation theory throughout this paper under the assumption that the category utility functions, v, g_1, and g_2, are linearly homogeneous. In the literature on aggregation theory, that assumption is called the "Santa Claus" hypothesis, since it equates the quantity aggregator function with the welfare function. If the category utility function is not

[7] While empirical results are not yet available for the augmented Divisia monetary aggregate, $\mathcal{M}(\mathbf{m}_t^a)$, extensive empirical results are available for the unaugmented Divisia monetary aggregates, $M(\mathbf{m}_t)$. See, e.g., Barnett (2012), Barnett and Chauvet (2011a,b), Barnett and Serletis (2000), Belongia and Ireland (20141,b,c), and Serletis and Gogas (2014).

linearly homogeneous, then the utility function, while still measuring welfare, is not the quantity aggregator function. The correct quantity aggregator function is then the distance function in microeconomic theory. While the utility function and the distance function both fully represent consumer preferences, the distance function, unlike the utility function, is always linearly homogenous. When normalized, the distance function is called the Malmquist index.

In the latter case, when welfare measurement and quantity aggregation are not equivalent, the Divisia index tracks the distance function, not the utility function, thereby continuing to measure the quantity aggregate, but not welfare. See Barnett (1987) and Caves, Christensen, and Diewert (1982). Hence the only substantive assumption in quantity aggregation is blockwise weak separability of components. Without that assumption there cannot exist an aggregate to track.

6.3. Discrete Time Approximation to the Divisia Index

If $(\mathbf{m}_t, \mathbf{c}_t)$ is acquired by maximizing (9) subject to (10) at instant of time *t*, then $v(\mathbf{m}_t, \mathbf{c}_t)$ is the exact augmented monetary services aggregate, $\mathcal{M}_t$, as written in equation (11). In continuous time, $\mathcal{M}_t = v(\mathbf{m}_t, \mathbf{c}_t)$ can be tracked without error by the Divisia index, which provides $\mathcal{M}_t$ as the solution to the differential equation

$$\frac{dlog\ \mathcal{M}_t}{dt} = \sum_{i=1}^{n} \omega_{it} \frac{dlog\ m_{it}}{dt} + \sum_{j=1}^{k} \widetilde{\omega}_{jt} \frac{dlog\ c_{jt}}{dt}\ , \qquad (26)$$

in accordance with equation (25). The share ω_{it} is the expenditure share of monetary asset *i* in the total services of monetary assets and credit cards at instant of time *t*,

$$\omega_{it} = \pi_{it} m_{it} / (\boldsymbol{\pi}_t' \mathbf{m}_t + \widetilde{\boldsymbol{\pi}}_t' \mathbf{c}_t),$$

while the share $\widetilde{\omega}_{it}$ is the expenditure share of credit card services, *i*, in the total services of monetary assets and credit cards at instant of time *t*,

$$\widetilde{\omega}_{it} = \tilde{\pi}_{it} c_{it} / (\boldsymbol{\pi}_t' \mathbf{m}_t + \tilde{\boldsymbol{\pi}}_t' \mathbf{c}_t).$$

Note that the time path of $(\mathbf{m}_t, \mathbf{c}_t)$ must continually maximize (9) subject to (10), in order for (26) to hold.

In discrete time, however, many different approximations to (25) are possible, because ω_{it} and $\widetilde{\omega}_{it}$ need not be constant during any given time interval. By far the most common discrete time approximations to the Divisia index is the Törnqvist-Theil approximation (often called the Törnqvist (1936) index or just the Divisia index in discrete time). That index can be viewed as the Simpson's rule approximation, where t is the discrete time period, rather than an instant of time:

$$\begin{aligned} log\ \mathcal{M}(\mathbf{m}_t^a) - log\ \mathcal{M}(\mathbf{m}_{t-1}^a) \\ = \sum_{i=1}^{n} \bar{\omega}_{it} \left(\log m_{it} - \log m_{i,t-1}\right) \\ + \sum_{i=1}^{k} \bar{\tilde{\omega}}_{it} \left(\log c_{it} - \log c_{i,t-1}\right), \end{aligned} \tag{27}$$

where $\bar{\omega}_{it} = (\omega_{it} + \omega_{i,t-1})/2$ and $\bar{\tilde{\omega}}_{it} = (\tilde{\omega}_{it} + \tilde{\omega}_{i,t-1})/2$.

A compelling reason exists for using the Törnqvist index as the discrete time approximation to the Divisia index. Diewert (1976) has defined a class of index numbers, called "superlative" index numbers, which have particular appeal in producing discrete time approximations to aggregator functions. Diewert defines a superlative index number to be one that is exactly correct for some quadratic approximation to the aggregator function, and thereby provides a second order local approximation to the unknown aggregator function. In this case the aggregator function is $\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t) = v(\mathbf{m}_t, \mathbf{c}_t)$. The Törnqvist discrete time approximation to the continuous time Divisia index is in the superlative class, because it is exact for the translog specification for the aggregator function. The translog is quadratic in the logarithms. If the translog specification is not exactly correct, then the discrete

Divisia index (27) has a third-order remainder term in the changes, since quadratic approximations possess third-order remainder terms.

With weekly or monthly monetary asset data, the Divisia monetary index, consisting of the first term on the right hand side of (27), has been shown by Barnett (1980) to be accurate to within three decimal places in measuring log changes in $M_t = M(\mathbf{m}_t)$ in discrete time. That three decimal place error is smaller than the roundoff error in the Federal Reserve's component data. We can reasonably expect the same to be true for our augments Divisia monetary index, (27), in measuring the log change of $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$.

7. Risk Adjustment

In index number theory, it is known that uncertainty about future variables have no effect on contemporaneous aggregates or index numbers, if preferences are intertemporally separable. Only contemporaneous risk is relevant. See, e.g., Barnett (1995). Prior to Barnett, Liu, and Jensen (1997)), the literature on index number theory assumed that contemporaneous prices are known with certainty, as is reasonable for consumer goods. But Poterba and Rotemberg (1987) observed that contemporaneous user cost prices of monetary assets are not known with certainty, since interest rates are not paid in advance. As a result, the need existed to extend the field of index number theory to the case of contemporaneous risk.

For example, the derivation of the Divisia index in Section 6.1 uses the perfect certainty first-order conditions for expenditure constrained maximization of $\mathcal{M}$, in a manner similar to Francois Divisia's (1925, 1926) derivation of the Divisia index for consumer goods. But if the contemporaneous user costs are not known with certainty, those first order conditions become Euler equations. This observation motivated Barnett, Liu, and Jensen (1997)) to repeat the steps in the Section 6.1 derivation with the first order conditions replaced by Euler equations. In this section, we analogously derive an extended augmented Divisia index using the Euler equations that apply under risk, with utility assumed to be intertemporally strongly

separable. The result is a Divisia index with the user costs adjusted for risk in a manner consistent with the CCAPM (consumption capital asset price model).[8]

The approach to our derivation of the extended index closely parallels that in Barnett, Liu, and Jensen (1997), Barnett and Serletis (2000, ch. 12), and Barnett (2012, Appendix D) for monetary assets alone. But our results, including credit card services, are likely to result in substantially higher risk adjustments than the earlier results for monetary assets alone, since interest rates on credit card debt are much higher and much more volatile than on monetary assets.

7.1 The Decision

Define Y to be the consumer's survival set, assumed to be compact. The decision problem in this section will differ from the one in section 2 not only by introducing risk, but also by adopting an infinite planning horizon. The consumption possibility set, $S(s)$, for period s is the set of survivable points, $(\boldsymbol{m}_s, \boldsymbol{c}_s, \boldsymbol{x}_s, A_s)$ satisfying equation (2).

The benchmark asset A_s provides no services other than its yield, R_s. As a result, the benchmark asset does not enter the consumer's contemporaneous utility function. The asset is held only as a means of accumulating wealth. The consumer's subjective rate of time preference, ξ, is assumed to be constant. The single-period utility function, $u(\boldsymbol{m}_t, \boldsymbol{c}_t, \boldsymbol{x}_t)$, is assumed to be increasing and strictly quasi-concave.

The consumer's decision problem is the following.

Problem 1. Choose the deterministic point $(\boldsymbol{m}_t, \boldsymbol{c}_t, \boldsymbol{x}_t, A_t)$ and the stochastic process $(\boldsymbol{m}_s, \boldsymbol{c}_s, \boldsymbol{x}_s, A_s)$, $s = t+1, \dots, \infty$, to maximize

$$u(\boldsymbol{m}_t, \boldsymbol{c}_t, \boldsymbol{x}_t) + E_t[\sum_{s=t+1}^{\infty} \left(\frac{1}{1+\xi}\right)^{s-t} u(\boldsymbol{m}_s, \boldsymbol{c}_s, \boldsymbol{x}_s)], \tag{28}$$

[8] Regarding CCAPM, see Lucas (1978), Breeden (1979), and Cochrane (2000).

Subject to $(\boldsymbol{m}_s, \boldsymbol{c}_s, \boldsymbol{x}_s, A_s) \in S(s)$ for $s = t$, t+1, … , ∞, and also subject to the transversality condition

$$\lim_{s\to\infty} E_t \left(\frac{1}{1+\xi}\right)^{s-t} A_s = 0. \tag{29}$$

7.2 Existence of an Augmented Monetary Aggregate for the Consumer

We assume that the utility function, u, is blockwise weakly separable in $(\boldsymbol{m}_s, \boldsymbol{c}_s)$ and in $\boldsymbol{x}_s$. Hence, there exists an augmented monetary aggregator function, $\mathcal{M}$, consumer goods aggregator function, X, and utility functions, F and H, such that

$$u(\boldsymbol{m}_s, \boldsymbol{c}_s, \boldsymbol{x}_s) = F[\mathcal{M}(\boldsymbol{m}_s, \boldsymbol{c}_s), X(\boldsymbol{x}_s)]\,. \tag{30}$$

We define the utility function V by $V(\boldsymbol{m}_s, \boldsymbol{c}_s, X_s) = F[\mathcal{M}(\boldsymbol{m}_s, \boldsymbol{c}_s), X_s]$, where aggregate consumption of goods is defined by $X_s = X(\boldsymbol{x}_s)$. It follows that the exact augmented monetary aggregate is

$$\mathcal{M}_s = \mathcal{M}(\boldsymbol{m}_s, \boldsymbol{c}_s). \tag{31}$$

The fact that blockwise weak separability is a necessary condition for exact aggregation is well known in the perfect-certainty case. If the resulting aggregator function also is linearly homogeneous, two-stage budgeting can be used to prove that the consumer behaves as if the exact aggregate were an elementary good, as in section 5.2. Although two-stage budgeting theory is not applicable under risk, $\mathcal{M}(\boldsymbol{m}_s, \boldsymbol{c}_s)$ remains the exact aggregation-theoretic quantity aggregate in a well-defined sense, even under risk.[9]

The Euler equations that will be of the most use to us below are those for monetary assets and credit card services. Those Euler equations are

[9] See Barnett (1995) and the appendix in Barnett, Liu, and Jensen (1997).

$$E_s\left[\frac{\partial V}{\partial m_{is}}-\rho\frac{p_s^*(R_s-r_{is})}{p_{s+1}^*}\frac{\partial V}{\partial X_{s+1}}\right]=0 \qquad (32a)$$

and

$$E_s\left[\frac{\partial V}{\partial c_{js}}-\rho\frac{p_s^*(e_{js}-R_s)}{p_{s+1}^*}\frac{\partial V}{\partial X_{s+1}}\right]=0 \qquad (32b)$$

for all $s \geq t, i = 1, \ldots, n$, and $j = 1, \ldots, k$, where $\rho = 1/(1+\xi)$ and where p_s^* is the exact price aggregate that is dual to the consumer goods quantity aggregate X_s.

Similarly, we can acquire the Euler equation for the consumer goods aggregate, X_s, rather than for each of its components. The resulting Euler equation for X_s is

$$E_s\left[\frac{\partial V}{\partial X_s}-\rho\frac{p_s^*(1+R_s)}{p_{s+1}^*}\frac{\partial V}{\partial X_{s+1}}\right]=0. \qquad (32c)$$

For the two available approaches to derivation of the Euler equations, see the Appendix.

7.3 The Perfect-Certainty Case

In the perfect-certainty case with finite planning horizon, we have already shown in section 2 that the contemporaneous nominal user cost of the services of m_{it} is equation (5) and the contemporaneous nominal user cost of credit card services is equation (7). We have also shown in section 6 that the solution value of the exact monetary aggregate, $\mathcal{M}(\boldsymbol{m}_t, \boldsymbol{c}_t) = \mathcal{M}(\boldsymbol{m}_{\mathrm{t}}^{\mathrm{a}})$, can be tracked without error in continuous time by the Divisia index, equation (25).

The flawless tracking ability of the index in the perfect-certainty case holds regardless of the form of the unknown aggregator function, $\mathcal{M}$. Aggregation results derived with finite planning horizon also hold in the limit with infinite planning horizon. See Barnett (1987, section 2.2). Hence those results continue

to apply. However, under risk, the ability of equation (25) to track $\mathcal{M}(\boldsymbol{m}_t, \boldsymbol{c}_t)$ is compromised.

7.4 New Generalized Augmented Divisia Index

7.4.1 User Cost Under Risk Aversion

We now find the formula for the user costs of monetary services and credit card services under risk.

Definition 1. *The contemporaneous risk-adjusted real user cost price of the services of* m_{it}^a *is* $\wp_{it}^a$*, defined such that*

$$\wp_{it}^a = \frac{\frac{\partial V}{\partial m_{it}^a}}{\frac{\partial V}{\partial X_t}}, i = 1,2, \dots, n+k.$$

The above definition for the contemporaneous user cost states that the real user cost price of an augmented monetary asset is the marginal rate of substitution between that asset and consumer goods.

For notational convenience, we convert the nominal rates of return, r_{it}, e_{jt} and R_t, to real total rates, $1 + r_{it}^*$, $1 + e_{jt}^*$ and $1 + R_t^*$ such that

$$1 + r_{it}^* = \frac{p_t^*(1 + r_{it})}{p_{t+1}^*}, \tag{33a}$$

$$1 + e_{jt}^* = \frac{p_t^*(1 + e_{jt})}{p_{t+1}^*}, \tag{33b}$$

$$1 + R_t^* = \frac{p_t^*(1 + R_t)}{p_{t+1}^*}, \tag{33c}$$

where r_{it}^*, e_{jt}^*, and R_t^* are called the real rates of excess return. Under this change of variables and observing that current-period marginal utilities are known with certainty, Euler equations (32a), (32b) and (32c) become

$$\frac{\partial V}{\partial m_{it}} - \rho E_t \left[(R_t^* - r_{it}^*) \frac{\partial V}{\partial X_{t+1}} \right] = 0, \tag{34}$$

$$\frac{\partial V}{\partial c_{jt}} - \rho E_t \left[\left(e_{jt}^* - R_t^* \right) \frac{\partial V}{\partial X_{t+1}} \right] = 0, \tag{35}$$

and

$$\frac{\partial V}{\partial X_t} - \rho E_t \left[(1 + R_t^*) \frac{\partial V}{\partial X_{t+1}} \right] = 0. \tag{36}$$

We now can provide our user cost theorem under risk.

Theorem 1 (a). *The risk adjusted real user cost of the services of monetary asset i under risk is* $\wp_{it}^m = \pi_{it} + \psi_{it}$, *where*

$$\pi_{it} = \frac{E_t R_t^* - E_t r_{it}^*}{1 + E_t R_t} \tag{37}$$

and

$$\psi_{it} = \rho(1 - \pi_{it}) \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}} \right)}{\frac{\partial V}{\partial X_t}} - \rho \frac{Cov\left(r_{it}^*, \frac{\partial V}{\partial X_{t+1}} \right)}{\frac{\partial V}{\partial X_t}}. \tag{38}$$

(b). *The risk adjusted real user cost of the services of credit card type j under risk is $\wp_{jt}^c = \tilde{\pi}_{jt} + \tilde{\psi}_{jt}$, where*

$$\tilde{\pi}_{jt} = \frac{E_t e_{jt}^* - E_t R_t^*}{1 + E_t R_t} \tag{39}$$

and

$$\tilde{\psi}_{jt} = \rho \frac{Cov\left(e_{jt}^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\left(1 + \tilde{\pi}_{jt}\right) \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} . \tag{40}$$

Proof. See the Appendix. ■

Under risk neutrality, the covariances in (38) and (40) would all be zero, because the utility function would be linear in consumption. Hence, the user cost of monetary assets and credit card services would reduce to $\pi_{i,t}$ and $\tilde{\pi}_{j,t}$ respectively, as defined in equation (37) and (39). The following corollary is immediate.

Corollary 1 to Theorem 1. Under risk neutrality, the user cost formulas are the same as equation (5) and (7) in the perfect-certainty case, but with all interest rates replaced by their expectations.

7.4.2 Generalized Augmented Divisia Index Under Risk Aversion

In the case of risk aversion, the first-order conditions are Euler equations. We now use those Euler equations to derive a generalized Divisia index, as follows.

Theorem 2. *In the share equations, $\omega_{it} = \pi_{it}^a m_{it}^a / \boldsymbol{\pi}_t^{a\prime} \mathbf{m}_t^a$, we replace the user costs, $\boldsymbol{\pi}_t^a = (\boldsymbol{\pi}_t', \tilde{\boldsymbol{\pi}}_t')'$, defined by (5) and (7), by the risk-adjusted user costs, $\wp_{it}^a$, defined by Definition 1, to produce the risk adjusted shares, $s_{it} = \wp_{it}^a m_{it}^a / \sum_{j=1}^{n+k} \wp_{jt}^a m_{jt}^a$. Under our*

weak-separability assumption, $V(\boldsymbol{m}_s, \boldsymbol{c}_s, X_s) = F[\mathcal{M}(\boldsymbol{m}_s, \boldsymbol{c}_s), X_s]$*, and our assumption that the monetary aggregator function* $\mathcal{M}$ *is linearly homogeneous, the following generalized augmented Divisia index is true under risk:*

$$dlog\ \mathcal{M}_t = \sum_{i=1}^{n+k} s_{it} dlog\ m_{it}^a. \tag{41}$$

Proof. See the Appendix. ■

The exact tracking of the Divisia monetary index is not compromised by risk aversion, as long as the adjusted user costs $\pi_{it} + \psi_{it}$ and $\tilde{\pi}_{jt} + \tilde{\psi}_{jt}$ are used in computing the index. The adjusted user costs reduce to the usual user costs in the case of perfect certainty, and our generalized Divisia index (41) reduces to the usual Divisia index (25). Similarly, the risk-neutral case is acquired as the special case with $\psi_{it} = \tilde{\psi}_{jt} = 0$, so that equations (37) and (39) serve as the user costs. In short, our generalized augmented Divisia index (41) is a true generalization in the sense that the risk-neutral and perfect-certainty cases are strictly nested special cases. Formally, that conclusion is the following.

Corollary 1 to Theorem 2. *Under risk neutrality, the generalized Divisia index (41) reduces to (25), where the user costs in the formula are defined by (37) and (39).*

7.5 CCAPM Special Case

As a means of illustrating the nature of the risk adjustments, $\psi_{i,t}$ and $\tilde{\psi}_{j,t}$, we consider a special case, based on the usual assumptions in CAPM theory of either quadratic utility or Gaussian stochastic processes. Direct empirical use of Theorems 1and 2, without any CAPM simplifications, would require availability of prior econometric estimates of the parameters of the utility function V and of the subjective rate of time discount. Under the usual CAPM assumptions, we show in this section that empirical use of Theorems 1 and 2 would require prior estimation

of only one property of the utility function: the degree of risk aversion, on which a large body of published information is available.

Consider first the following case of utility that is quadratic in consumption of goods, conditionally on the level of monetary asset and credit card services.

Assumption 1. Let V have the form

$$V(\boldsymbol{m}_t, \boldsymbol{c}_t, X_t) = F[\mathcal{M}(\boldsymbol{m}_t, \boldsymbol{c}_t), X_t] = A[\mathcal{M}(\boldsymbol{m}_t, \boldsymbol{c}_t)]X_t - \frac{1}{2}B[\mathcal{M}(\boldsymbol{m}_t, \boldsymbol{c}_t)]X_t^2, \quad (42)$$

where A is a positive, increasing, concave function and B is a nonnegative, decreasing, convex function.

The alternative assumption is Guassianity, as follows:

Assumption 2. Let $\left(r_{it}^*, e_{jt}^*, X_{t+1}\right)$ be a trivariate Gaussian process for each asset $i = 1, \dots, n$, and credit card service, $j = 1, \dots, k$.

We also make the following conventional CAPM assumption:

Assumption 3. The benchmark rate process is deterministic or already risk-adjusted, so that R_t^* is the risk-free rate.

Under this assumption, it follows that

$$Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) = 0.$$

We define $H_{t+1} = H(\mathcal{M}_{t+1}, X_{t+1})$ to be the well-known Arrow-Pratt measure of absolute risk aversion,

$$H(\mathcal{M}_{t+1}, X_{t+1}) = -\frac{E_t[V'']}{E_t[V']}, \tag{43}$$

where $V' = \partial V(\boldsymbol{m}_{t+1}^a, X_{t+1})/\partial X_{t+1}$ and $V'' = \partial^2 V(\boldsymbol{m}_{t+1}^a, X_{t+1})/\partial X_{t+1}^2$. In this definition, risk aversion is measured relative to consumption risk, conditionally upon the level of augmented monetary services produced by $\mathcal{M}_{t+1} = \mathcal{M}(\boldsymbol{m}_t, \boldsymbol{c}_t)$. Under risk aversion, H_{t+1} is positive and increasing in the degree of absolute risk aversion. The following lemma is central to our Theorem 3.

Lemma 2. *Under Assumption 3 and either Assumption 1 or Assumption 2, the user-cost risk adjustments, ψ_{it} and $\tilde{\psi}_{jt}$, defined by (38) and (40), reduce to*

$$\psi_{it} = \frac{1}{1+R_t^*} H_{t+1} cov(r_{it}^*, X_{t+1}) \tag{44a}$$

and

$$\tilde{\psi}_{jt} = -\frac{1}{1+R_t^*} H_{t+1} cov\left(e_{jt}^*, X_{t+1}\right). \tag{44b}$$

Proof. See the Appendix. ■

The following theorem identifies the effect of the risk adjustment on the expected own interest rates in the user cost formulas.

Theorem 3. *Let $\hat{H}_t \quad H_{t+1} X_t$. Under the assumptions of Lemma 2, we have the following for each asset $i = 1, \ldots, n$, and credit card service, $j = 1, \ldots, k$.*

$$\wp_{it}^m = \frac{E_t R_t^* - (E_t r_{it}^* - \phi_{it})}{1 + E_t R_t^*}, \tag{45}$$

where

$$\phi_{it} = \hat{H}_t \, Cov\left(r_{it}^*, \frac{X_{t+1}}{X_t}\right), \tag{46}$$

and

$$\wp_{jt}^c = \frac{(E_t e_{jt}^* - \tilde{\phi}_{jt}) - E_t R_t^*}{1 + E_t R_t^*}, \tag{47}$$

where

$$\tilde{\phi}_{jt} = \hat{H}_t \, Cov\left(e_{jt}^*, \frac{X_{t+1}}{X_t}\right). \tag{48}$$

Proof. See the Appendix. ■

As defined, $\hat{H}_t$ is a time shifted Arrow-Pratt relative risk aversion measure. Theorem 3 shows that the risk adjustment on the own interest rate for a monetary asset or credit card service depends upon relative risk aversion, $\hat{H}_t$, and the covariance between the consumption growth path, X_{t+1}/X_t, and the real rate of excess return earned on a monetary asset, r_{it}^*, or paid on a credit card service, e_{jt}^*.

7.6 Magnitude of the Adjustment

In accordance with the large and growing literature on the equity premium puzzle, the CCAPM risk adjustment term is widely believed to be biased downward.[10] A promising explanation may be the customary assumption of intertemporal separability of utility, since response to a change in an interest rate may not be fully reflected in contemporaneous changes in consumption. Hence the contemporaneous covariance in the CCAPM "beta" correction may not take full account of the effect of an interest rate change on life style. An approach to risk adjustment without assumption of intertemporal separability was developed for monetary aggregation by Barnett and Wu (2005).

[10] See, e.g., Campbell and Cochrane (1999), Cochrane (2000), Kocherlakota (1996), Marshall (1997), Mehra and Prescott (1985).

8. Conclusions

Many economists have wondered how the transactions services of credit cards could be included in monetary aggregates. The conventional simple sum accounting approach precludes solving that problem, since accounting conventions do not permit adding liabilities to assets. But economic aggregation and index number theory measure service flows independently of whether from assets or liabilities. We have provided theory solving that long overlooked problem.

We have provided the solution under various levels of complexity in terms of theory, econometrics, and data availability. The most easily implemented approach is Method 1 in section 2.1 under risk neutrality. A theoretically more appealing approach is Method 2 in that section, also under risk neutrality. We have provided the CCAPM approach to risk adjustment. A more demanding approach would remove the CCAPM assumption of intertemporal separability, as derived for monetary aggregation by Barnett and Wu (2005). Adapting that advanced approach to our augmented aggregates, including credit card services, remains a topic for future research. Hence, six possible approaches exist to incorporating credit card services into monetary aggregates: Method 1 or Method 2 under risk neutrality, Method 1 or Method 2 under CCAPM risk, or Method 1 or Method 2 under intertemporally nonseparable risk.

What remains to be determined from empirical implementation is the robustness of results across the six possible combinations of approaches, as would be needed to justify use of one of the less demanding approaches. While excluding credit card services, the currently available Divisia monetary aggregates have been found to be reasonably robust to introduction of risk, variations of the benchmark rate, introduction of taxation of interest rates, and other such refinements.[11] But

[11] While those refinements slightly change the unaugmented Divisia monetary aggregates, those changes are negligible relative to the gap between the simple sum monetary aggregate path and the corresponding Divisia monetary aggregate path. See, e.g., the online library of relevant research and the Divisia monetary aggregates databases at the Center for Financial Stability (www.centerforfinancialstability.org/amfm.php).

such simplifications might not be the case with our augmented monetary aggregates, because of the high and volatile interest rates on credit card balances.

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APPENDIX

(I) Derivation of the User Cost Formula for Credit Card Services, Equation (7), in the Infinite Lifetimes Case under Perfect Certainty:

From equation 2, the flow of funds identities, for $s \quad t, t+1, \ldots, \infty,$ are

$$\begin{aligned}\mathbf{p}_s'\mathbf{x}_s = w_s L_s &+ \sum_{i=1}^{n}\left[(1+r_{i,s-1})p_{s-1}^* m_{i,s-1} - p_s^* m_{is}\right] \\ &+ \sum_{j=1}^{k}\left[p_s^* c_{js} - (1+e_{j,s-1})p_{s-1}^* c_{j,s-1}\right] \\ &+ \left[(1+R_{s-1})p_{s-1}^* A_{s-1} - p_s^* A_s\right]. \qquad \text{(A. 1)}\end{aligned}$$

The intertemporal utility function is

$$u(\boldsymbol{m}_t, \boldsymbol{c}_t, \boldsymbol{x}_t) + E_t\left[\sum_{s=t+1}^{\infty}\left(\frac{1}{1+\xi}\right)^{s-t} u(\boldsymbol{m}_s, \boldsymbol{c}_s, \boldsymbol{x}_s)\right]. \qquad \text{(A. 2)}$$

Let $\Im$ be the Lagrangian for maximizing intertemporal utility subject to the sequence of flow of funds identities for $s \quad t, \ldots, \infty$, and let λ_t be the Lagrange multiplier for the *t*'th constraint. Then the following are the first order conditions for maximizing (A.2) subject to the sequence of constraints, (A.1).

$$\frac{\partial \Im}{\partial A_t} \quad -\lambda_{t+1}(1+R_t)p_t^* + \lambda_t p_t^* \quad 0, \qquad \text{(A.3)}$$

$$\frac{\partial \Im}{\partial x_{it}} \quad \frac{\partial u}{\partial x_{it}} = -\lambda_t p_{it} \quad 0, \qquad \text{(A.4)}$$

$$\frac{\partial \Im}{\partial m_{it}} \quad \frac{\partial u}{\partial m_{it}} \quad \lambda_t p_t^* + \lambda_{t+1}(1+r_{it})p_t^* \quad 0, \qquad \text{(A.5)}$$

$$\frac{\partial \Im}{\partial c_{jt}} \quad \frac{\partial u}{\partial c_{jt}} + \lambda_t p_t^* - \lambda_{t+1}(1+e_{jt}) p_t^* \quad 0. \tag{A.6}$$

From equation (A.3), we have

$$-\lambda_{t+1}(1+R_t) + \lambda_t \quad 0. \tag{A.7}$$

Substitute equation (A.7) into (A.6) to eliminate λ_{t+1}, we get

$$\frac{\partial u}{\partial c_{jt}} \quad - \lambda_t p_t^* + \frac{\lambda_t}{1+R_t} p_t^* (1+e_{jt}). \tag{A.8}$$

Rearranging we get the first order condition that identifies $\tilde{\pi}_{jt}$ as the user cost price of credit card services:

$$\frac{\partial u}{\partial c_{jt}} \quad \lambda_t \tilde{\pi}_{jt}, \tag{A.9}$$

where

$$\tilde{\pi}_{jt} \quad p_t^* \frac{e_{jt} - R_t}{1+R_t}. \qquad \blacksquare \tag{A.10}$$

(II) Derivation of Euler Equations for Credit Card Services, Equation (35):

The following are the Euler equations provided in the paper as equations (34), (35), and (36):

$$\frac{\partial V}{\partial m_{it}} - \rho E_t \left[(R_t^* - r_{it}^*) \frac{\partial V}{\partial X_{t+1}} \right] = 0, \tag{A. 11}$$

$$\frac{\partial V}{\partial c_{jt}} - \rho E_t \left[\left(e_{jt}^* - R_t^* \right) \frac{\partial V}{\partial X_{t+1}} \right] = 0, \tag{A. 12}$$

$$\frac{\partial V}{\partial X_t} - \rho E_t \left[(1 + R_t^*) \frac{\partial V}{\partial X_{t+1}} \right] = 0. \tag{A.13}$$

for all $s \geq t, i = 1, \ldots, n$, and $j = 1, \ldots, k$, where $\rho = 1/(1 + \xi)$ and where p_s^* is the exact price aggregate that is dual to the consumer goods quantity aggregate X_s.

Equation (A.11) was derived in Barnett (1995, Sec 2.3) using Bellman's method. An alternative approach to that derivation using calculus of variations was provided by Poterba and Rotemberg (1987). Equation (A.12) follows by the same approach to derivation, using either Bellman's method or calculus of variations. We are not providing the lengthy derivation of (A.12) in this appendix, since the steps in the Bellman method approach for this class of models are provided in detail in Barnett and Serletis (2000, pp. 201-204).

(III) Proof of Theorem 1

Theorem 1 (a). *The risk adjusted real user cost of the services of monetary asset i under risk is* $\wp_{it}^m = \pi_{it} + \psi_{it}$, *where*

$$\pi_{it} = \frac{E_t R_t^* - E_t r_{it}^*}{1 + E_t R_t^*} \tag{A.14}$$

and

$$\psi_{it} = \rho(1 - \pi_{it}) \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho \frac{Cov\left(r_{it}^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}. \tag{A.15}$$

(b). *The risk adjusted real user cost of the services of credit card type j under risk is* $\wp_{jt}^c = \tilde{\pi}_{jt} + \tilde{\psi}_{jt}$, *where*

$$\tilde{\pi}_{jt} = \frac{E_t e_{jt}^* - E_t R_t^*}{1 + E_t R_t^*} \tag{A. 16}$$

and

$$\tilde{\psi}_{jt} = \rho \frac{Cov\left(e_{jt}^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\left(1 + \tilde{\pi}_{jt}\right) \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} . \tag{A. 17}$$

Proof. For the analogous proof in the case of monetary assets only, relevant to part (a), see Barnett, Liu, and Jensen (1997), Barnett and Serletis (2000, ch. 12), or Barnett (2012, Appendix D). We provide the proof of part (b) for the extended case including credit. There are two approaches to proving this important theorem, the direct approach and the indirect approach. We provide both approaches, beginning with the indirect approach.

By definition (1) in the paper, we have for the credit card services user cost price

$$\wp_{jt}^c \quad \frac{\frac{\partial V}{\partial c_{jt}}}{\frac{\partial V}{\partial X_t}} . \tag{A.18}$$

Defining $\tilde{\psi}_{jt}$ to be $\tilde{\psi}_{jt} = \wp_{jt}^c - \tilde{\pi}_{jt}$, it follows that

$$\frac{\partial V}{\partial c_{jt}} \quad (\tilde{\pi}_{jt} + \tilde{\psi}_{jt}) \frac{\partial V}{\partial X_t} .$$

Substituting equations (A.12) and (A.13) into this equation, we get

$$\rho E_t \left[e_{jt}^* - R_t^* \frac{\partial V}{\partial X_{t+1}} \right] \quad (\tilde{\pi}_{jt} + \tilde{\psi}_{jt}) \rho E_t \left[1 + R_t^* \frac{\partial V}{\partial X_{t+1}} \right] .$$

Using the expectation of the product of correlated random variables, we have

$$E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$\left\{\left[\frac{E_t e^*_{jt}-E_t R^*_t}{1+E_t R^*_t}\right]+\tilde{\psi}_{jt}\right\}\left\{E_t\left[1+R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+Cov\left(R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)\right\}.$$

Multiplying $1+E_t R^*_t$ through on both sides of the equation, we get:

$$\left[1+E_t R^*_t\right]E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+\left[1+E_t R^*_t\right]Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$\left[E_t\left[e^*_{jt}-R^*_t\right]+\left[1+E_t R^*_t\right]\tilde{\psi}_{jt}\right]\left\{E_t\left[1+R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+Cov\left(R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)\right\}.$$

Manipulating the algebra, we have

$$E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_t R^*_t\,E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$+E_t R^*_t\,Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$\left[E_t\left[e^*_{jt}-R^*_t\right]+\left[1+E_t R^*_t\right]\tilde{\psi}_{jt}\right]\left\{E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_t R^*_t\,E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+Cov\left(R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)\right\},$$

and hence

$$E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_tR^*_t\,E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$+E_tR^*_t\,Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_t\left[e^*_{jt}-R^*_t\right]E_tR^*_t\left(E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)\right)+E_t\left[e^*_{jt}-R^*_t\right]Cov\left(R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$+\left(1+E_tR^*_t\right)\tilde{\psi}_{jt}\left\{E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_tR^*_t\left(E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)\right)+Cov\left(R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)\right\}.$$

Notice that by equation (A.13),

$$\frac{\partial V}{\partial X_t}=\rho E_t\left[\left(1+R^*_t\right)\frac{\partial V}{\partial X_{t+1}}\right]$$
$$=\rho\left\{E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_tR^*_t\left(E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)\right)+Cov\left(R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)\right\}.$$

Substituting this back into the prior equation, we have

$$E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_tR^*_t\,E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$+E_tR^*_t\,Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$E_t\left[e^*_{jt}-R^*_t\right]E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_t\left[e^*_{jt}-R^*_t\right]E_tR^*_t\left(E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)\right)+E_t\left[e^*_{jt}-R^*_t\right]Cov\left(R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$+\left(1+E_tR^*_t\right)\tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right).$$

Simplifying the equation, we get

$$Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)+E_tR^*_t\,Cov\left(e^*_{jt}-R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$E_t\left[e^*_{jt}-R^*_t\right]Cov\left(R^*_t,\frac{\partial V}{\partial X_{t+1}}\right)+\left(1+E_tR^*_t\right)\tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right).$$

Recall that by equation (A.16),

$$\tilde{\pi}_{jt} = \frac{E_t e^*_{jt} - E_t R^*_t}{1 + E_t R^*_t}.$$

Substituting this equation back into the prior equation, we have

$$Cov\left(e^*_{jt} - R^*_t, \frac{\partial V}{\partial X_{t+1}}\right) + E_t R^*_t \; Cov\left(e^*_{jt} - R^*_t, \frac{\partial V}{\partial X_{t+1}}\right)$$
$$\tilde{\pi}_{jt}\left(1 + E_t R^*_t\right) Cov\left(R^*_t, \frac{\partial V}{\partial X_{t+1}}\right) + \left(1 + E_t R^*_t\right) \tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right).$$

Rearranging the equation, we have

$$\left(1 + E_t R^*_t\right) Cov\left(e^*_{jt} - R^*_t, \frac{\partial V}{\partial X_{t+1}}\right) \tilde{\pi}_{jt}\left(1 + E_t R^*_t\right) Cov\left(R^*_t, \frac{\partial V}{\partial X_{t+1}}\right) + \left(1 + E_t R^*_t\right) \tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right),$$

so that

$$Cov\left(e^*_{jt} - R^*_t, \frac{\partial V}{\partial X_{t+1}}\right) \tilde{\pi}_{jt} Cov\left(R^*_t, \frac{\partial V}{\partial X_{t+1}}\right) + \tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right).$$

Hence, it follows that

$$\tilde{\psi}_{jt} \rho \frac{Cov\left(e^*_{jt} - R^*_t, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} \rho\tilde{\pi}_{jt} \frac{Cov\left(R^*_t, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}$$
$$\rho \frac{Cov\left(e^*_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} \rho \frac{Cov\left(R^*_t, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\tilde{\pi}_{jt} \frac{Cov\left(R^*_t, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}$$
$$\rho \frac{Cov\left(e^*_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\left(1 + \tilde{\pi}_{jt}\right) \frac{Cov\left(R^*_t, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}.$$

The alternative direct approach to proof is the following.

By equation (A.13), we have

$$\frac{\partial V}{\partial X_t} \quad \rho E_t\left[\; 1+ \; R_t^* \; \frac{\partial V}{\partial X_{t+1}} \right]$$

$$\rho \; 1+ \; E_t R_t^* \; \left(E_t\left(\frac{\partial V}{\partial X_{t+1}} \right) \right) + \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}} \right).$$

Rearranging, we get

$$\rho \; 1+E_t R_t^* \; \left(E_t\left(\frac{\partial V}{\partial X_{t+1}} \right) \right) \quad \frac{\partial V}{\partial X_t} - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}} \right),$$

and hence

$$\rho E_t\left(\frac{\partial V}{\partial X_{t+1}} \right) \quad \frac{1}{1+E_t R_t^*}\left[\frac{\partial V}{\partial X_t} - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}} \right) \right]. \qquad \text{(A.19)}$$

But from (A.12), we have

$$\frac{\partial V}{\partial c_{jt}} \quad \rho E_t\left[\; e_{jt} - R_t^* \; \frac{\partial V}{\partial X_{t+1}} \right].$$

From the expectation of the correlated product, we then have

$$\frac{\partial V}{\partial c_{jt}} \quad \rho E_t \; e_{jt} - R_t^* \; E_t\left(\frac{\partial V}{\partial X_{t+1}} \right) + \rho Cov\left[\; e_{jt} - R_t^* \; , \frac{\partial V}{\partial X_{t+1}} \right],$$

so that

$$\frac{\partial V}{\partial c_{jt}} \quad \rho E_t \; e_{jt} - R_t^* \; E_t\left(\frac{\partial V}{\partial X_{t+1}} \right) + \rho Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}} \right) - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}} \right). \qquad \text{(A.20)}$$

Now substitute equation (A.19) into equation (A.20), to acquire

$$\frac{\partial V}{\partial c_{jt}} \quad \frac{E_t\ e_{jt} - R_t^*}{1 + E_t R_t^*}\left[\frac{\partial V}{\partial X_t} - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)\right] + \rho Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right) - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)$$
$$\tilde{\pi}_{jt}\left[\frac{\partial V}{\partial X_t} - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)\right] + \rho Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right) - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right).$$

Multiplying and dividing the right side by $\frac{\partial V}{\partial X_t}$, we get

$$\frac{\partial V}{\partial c_{jt}} \quad \frac{\partial V}{\partial X_t}\left\{\tilde{\pi}_{jt} - \rho\tilde{\pi}_{jt}\frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} + \rho\frac{Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}\right\}$$
$$\frac{\partial V}{\partial X_t}\left\{\tilde{\pi}_{jt} + \rho\frac{Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\ (1 + \tilde{\pi}_{jt})\ \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}\right\}.$$

Define $\tilde{\psi}_{jt}$ by

$$\tilde{\psi}_{jt} \quad \rho\frac{Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\ (1 + \tilde{\pi}_{jt})\ \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}.$$

Then we have

$$\frac{\frac{\partial V}{\partial c_{jt}}}{\frac{\partial V}{\partial X_t}} \quad \tilde{\pi}_{jt} + \tilde{\psi}_{jt},$$

so that

$$\wp^{c}_{jt} = \tilde{\pi}_{jt} + \tilde{\psi}_{jt}. \qquad \blacksquare$$

(IV) Proof of Lemma 2:

Assumption 1. Let V have the form

$$V(\boldsymbol{m}_t, \boldsymbol{c}_t, X_t) = F[\mathcal{M}(\boldsymbol{m}_t, \boldsymbol{c}_t), X_t] = A[\mathcal{M}(\boldsymbol{m}_t, \boldsymbol{c}_t)]X_t - \frac{1}{2}B[\mathcal{M}(\boldsymbol{m}_t, \boldsymbol{c}_t)]X_t^2, \quad \text{(A. 21)}$$

where A is a positive, increasing, concave function and B is a nonnegative, decreasing, convex function.

Assumption 2. Let $\left(r_{it}^*, e_{jt}^*, X_{t+1}\right)$ be a trivariate Gaussian process for each asset $i = 1, \dots, n$, and credit card service, $j = 1, \dots, k$.

Assumption 3. The benchmark rate process is deterministic or already risk-adjusted, so that R_t^* is the risk-free rate.

Under this assumption, it follows that

$$Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) = 0.$$

Define $H_{t+1} = H(\mathcal{M}_{t+1}, X_{t+1})$ to be the well-known Arrow-Pratt measure of absolute risk aversion,

$$H(\mathcal{M}_{t+1}, X_{t+1}) = -\frac{E_t[V'']}{E_t[V']}, \tag{A. 22}$$

where $V' = \partial V(\boldsymbol{m}^a_{t+1}, X_{t+1})/\partial X_{t+1}$ and $V'' = \partial^2 V(\boldsymbol{m}^a_{t+1}, X_{t+1})/\partial X^2_{t+1}$.

Lemma 2. *Under Assumption 3 and either Assumption 1 or Assumption 2, the user-cost risk adjustments, ψ_{it} and $\tilde{\psi}_{jt}$, defined by (A.15) and (A.17), reduce to*

$$\psi_{it} = \frac{1}{1+R^*_t} H_{t+1} cov(r^*_{it}, X_{t+1}) \tag{A. 23}$$

and

$$\tilde{\psi}_{jt} = -\frac{1}{1+R^*_t} H_{t+1} cov\left(e^*_{jt}, X_{t+1}\right). \tag{A. 24}$$

Proof. For the analogous proof in the case of monetary assets only, relevant to equation (44a), see Barnett, Liu, and Jensen (1997), Barnett and Serletis (2000, ch. 12), or Barnett (2012, Appendix D). We provide the proof of equation (A.24) for the extended case including credit.

Under Assumption 3, the benchmark asset is risk-free, so that

$$Cov\left(R^*_t, \frac{\partial V}{\partial X_{t+1}}\right) \quad 0.$$

By equation (A.17),

$$\tilde{\psi}_{jt} \quad \rho \frac{Cov\left(e^*_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} \quad \rho\ 1+\tilde{\pi}_{jt} \quad \frac{Cov\left(R^*_t, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}$$
$$\rho \frac{Cov\left(e^*_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}.$$

But by equation (A.13),

$$\frac{\partial V}{\partial X_t} \quad \rho E_t\left[\ 1+\ R^*_t\ \frac{\partial V}{\partial X_{t+1}}\right],$$

So

$$\tilde{\psi}_{jt} \quad \rho \frac{Cov\left(e^*_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\rho\ 1+R^*_t\ \ E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)}$$
$$\frac{Cov\left(e^*_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{1+R^*_t\ \ E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)}. \qquad \text{(A.25)}$$

Under Assumption 1,

$$\frac{\partial V}{\partial X_t} \quad A\left[\mathrm{M} \quad \boldsymbol{m}_t, \boldsymbol{c}_t\ \right] - B\left[\mathrm{M} \quad \boldsymbol{m}_t, \boldsymbol{c}_t\ \right] X_t\ .$$

Hence,

$$\frac{\partial^2 V}{\partial X_t^{\ 2}} \quad -B\left[\mathrm{M} \quad \boldsymbol{m}_t, \boldsymbol{c}_t\ \right].$$

Shifting one period forward, those two equations become

$$\frac{\partial V}{\partial X_{t+1}} \quad V' \quad A - BX_{t+1}$$

and

$$\frac{\partial^2 V}{\partial X_t^{\,2}} \quad V'' \quad -B\,.$$

Substituting into equation (A.25), we get

$$\begin{aligned}
\tilde{\psi}_{jt} \quad & \frac{Cov\ \ e_{jt}^{*},\ A - BX_{t+1}}{1 + R_t^{*}\ \ E_t\left(\dfrac{\partial V}{\partial X_{t+1}}\right)} \\
& \frac{-B}{1 + R_t^{*}} \frac{Cov\ \ e_{jt}^{*},\ X_{t+1}}{E_t\ \ V'} \\
& \frac{1}{1 + R_t^{*}} \frac{E\ \ V''}{E_t\ \ V'} Cov\ \ e_{jt}^{*},\ X_{t+1} \\
& - \frac{1}{1 + R_t^{*}} H_{t+1} Cov\ \ e_{jt}^{*},\ X_{t+1}\ \ .
\end{aligned}$$

Alternatively, consider Assumption 2. We then can use Stein's lemma, which says the following.[12] Suppose (*X*,*Y*) are multivariate normal. Then

$$Cov(g(X), Y) \quad E(g'(X))Cov(X, Y).$$

In that formula, let $g(X)$ $\frac{\partial V}{\partial X_{t+1}}$, X X_{t+1}, and Y e_{jt}^{*}. Then from Stein's lemma, we have

$$Cov\left(e_{jt}^{*}, \frac{\partial V}{\partial X_{t+1}}\right) \quad E_t\left(\frac{\partial^2 V}{\partial X_{t+1}^2}\right) Cov\ \ X_t, e_{jt}^{*}\ \ .$$

Substituting into (A.25), we get

[12] For Stein's lemma, see Stein (1973), Ingersoll (1987, p. 13, eq. 62) or Rubinstein (1976).

$$\tilde{\psi}_{jt} \quad \frac{E_t\left(\frac{\partial^2 V}{\partial X_{t+1}^2}\right) Cov\ \ X_t, e_{jt}^*}{1+R_t^*\ \ E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)}.$$

Using the definitions of V', V'', and H_{t+1}, we have

$$\tilde{\psi}_{jt} = -\frac{1}{1+R_t^*} H_{t+1} Cov\ \ e_{jt}^*,\ X_{t+1}\ \ . \qquad \blacksquare$$

(V) Proof of Theorem 3:

Theorem 3. *Let* $\hat{H}_t \quad H_{t+1} X_t$. *Under the assumptions of Lemma 2, we have the following for each asset* $i = 1, \dots, n$, *and credit card service,* $j = 1, \dots, k$,

$$\wp_{it}^{\ m} = \frac{E_t R_t^* - (E_t r_{it}^* - \phi_{it})}{1 + E_t R_t^*}, \tag{A. 26}$$

where

$$\phi_{it} = \hat{H}_t\, Cov\left(r_{it}^*, \frac{X_{t+1}}{X_t}\right), \tag{A. 27}$$

and

$$\wp_{jt}^{\ c} = \frac{(E_t e_{jt}^* - \tilde{\phi}_{jt}) - E_t R_t^*}{1 + E_t R_t^*}, \tag{A. 28}$$

where

$$\tilde{\phi}_{jt} = \hat{H}_t\, Cov\left(e_{jt}^*, \frac{X_{t+1}}{X_t}\right). \tag{A. 29}$$

Proof. For the proof in the case of monetary assets only, relevant to equations (A.26) and (A.27), see Barnett, Liu, and Jensen (1997), Barnett and Serletis (2000, ch. 12), or Barnett (2012, Appendix D). We here provide the proof of equations (A.28) and (A.29) for the extended case including credit.

From part b of Theorem 1,

$$\wp_{jt}^{c} \quad \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} + \tilde{\psi}_{jt} .$$

Letting $\hat{H}_t \quad H_{t+1} X_t$. and using Lemma 2, we get

$$\begin{aligned} \wp_{jt}^{c} \quad & \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} - \frac{H_{t+1} Cov\left(e_{jt}^{*},\, X_{t+1} \right)}{1 + E_t R_t^{*}} \\ & \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} - \frac{H_{t+1} X_t Cov\left(e_{jt}^{*},\, \frac{X_{t+1}}{X_t} \right)}{1 + E_t R_t^{*}} \\ & \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} - \frac{\hat{H}_t Cov\left(e_{jt}^{*},\, \frac{X_{t+1}}{X_t} \right)}{1 + E_t R_t^{*}} . \end{aligned}$$

Define $\tilde{\phi}_{j,t} \quad \hat{H}_t Cov\left(e_{jt}^{*},\, \frac{X_{t+1}}{X_t} \right)$ to get

$$\begin{aligned} \wp_{jt}^{c} \quad & \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} - \frac{\tilde{\phi}_{jt}}{1 + E_t R_t^{*}} \\ & \frac{(E_t e_{jt}^{*} - \tilde{\phi}_{jt}) - E_t R_t^{*}}{1 + E_t R_t^{*}} . \end{aligned}$$

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